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EXISTENCE, UNIQUENESS AND STABILITY SOLUTIONS FOR NEW NONLINEAR SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

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ABSTRACT

In this article, we established the existence, uniqueness and stability solutions for a nonlinear system of integro-differential equations of Volterra type in Banach spaces. Krasnoselskii Fixed point theorems and Picard approximation method are the main tool used here to establish the existence and uniqueness results.

Keywords: Existence and uniqueness, Picard approximation, Integrodifferential equations, Krasnoselskii fixed point theorem.

1. INTRODUCTION

The integral equations form an important part of applied mathematics, with links with many theoretical fields, especially with practical fields (K. E. Atkinson, 1997; P. Linz, 1985 ; E. Babolian and J. Biazar, 2000) The Volterra integral equations were introduced by Vito Volterra and then studied by Traian Lalescu in his 1908 thesis. Volterra integral equations find application in demography, the study of Viscoelastic materials, and in insurance mathematics through the renewal equation. Fredholm equations arise naturally in the theory of signal processing, most notably as the famous spectral concentration problem popularized by David Slepian. They also commonly arise in linear forward modelling and inverse problems. Throughout the last decade, physicists and mathematicians have paid attention to the concept of fractional calculus. Actually, real problems in scientific fields such as groundwater problems, physics, mechanics, chemistry, and biology are described by partial differential equations or integral equations. Many scholars have shown with great success the applications of fractional calculus to groundwater pollution and groundwater flow problems, acoustic wave problems, and others (F. Mainardi, 1997; P. Zhuang, F. Liu, V. Anh, and I. Turner, 2009; S. B. Yuste and L. Acedo, 2005; C.-M. Chen, F. Liu, I. Turner, and V. Anh, 2007).

The Picard iteration method, or the successive approximations method, is a direct and convenient technique for the resolution of differential equations. This method solves any problem by finding successive approximations to the solution by starting with the zeroth approximation. The symbolic computation applied to the Picard iteration is considered in (Parker, G.E. and Sochacki, J.S. 1996; Bailey, P.B., Shampine, L.F. and Waltman, P.E. 1968) and the Picard iteration can be used to generate the Taylor series solution for an ordinary differential equation (Butris, R. N. 1994; Butris R.N., 2015; Butris, R. N. and Ghada, Sh. J. 2006) studied existence and unique solution for different kind of equations and (Raad N. Butris and Hojeen M. Haji 2019) studied existence and unique solution of Volltera-friedholm:

$$\frac{dx}{dt} = Ax + f\left(t, \varepsilon, \int_{-\infty}^{t} R(t-\tau)\big(x(\tau) - y(\tau)\big)d\tau\right)$$

In this work our aim is to show the existence solutions of the system of integrodifferential equations:

$$\frac{dx}{dt} = Ax(t) + \int_{-\infty}^{t} K(t,s)F\big(t,s,x(s),y(s)\big)ds$$
$$\frac{dy}{dt} = By(t) + \int_{-\infty}^{t} G(t,s)H\big(t,s,x(s),y(s)\big)ds \qquad (1)$$

Where $x \in D \subseteq R^n$ and $y \in D_1 \subseteq R^n, D$ and $D_1$are closed and bounded, $T \in R$, let $F(t,s,x(t),y(t)), H(t,s,x(t),y(t))$ is defined on the domain:

$$D_* = \{(t,s,x,y) \epsilon\, [0,T] \text{ x } [0,T] \text{x } D \text{ x } D_1\} \quad (2)$$

Assume that the vector functions $F(t,s,x(t),y(t)), H(t,s,x(t),y(t))$, and kernels $K(t,s), G(t,s)$ are satisfying the following inequalities:

$$\|F(t,s,x(t),y(t))\| \le M_1 \ , \quad \|H(t,s,x(t),y(t))\| \le M_2 \quad (3)$$

$$\|F(t,s,x_2,y_2) - F(t,s,x_1,y_1)\| \le L_1(\|x_2 - x_1\| + \|y_2 - y_1\| \quad (4)$$

$$\|H(t,s,x_2,y_2) - H(t,s,x_1,y_1)\| \le L_2(\|x_2 - x_1\| + \|y_2 - y_1\| \quad (5)$$

$$\|K(t,s)\| \le \delta_1 e^{-\lambda_1(t-s)}, \|G(t,s)\| \le \delta_2 e^{-\lambda_2(t-s)} \quad (6)$$

$$\|e^{A(t-s)}\| \le Q_1 \quad , \|e^{B(t-s)}\| \le Q_2 \quad (7)$$

Where $L_1, L_2, M_1, M_2, \lambda_1, \lambda_2, \delta_1, \delta_2, Q_1, Q_2$ are positive constants
$x_1, x_2 \in D$ $y_1, y_2 \in D_1$

$t\,, s\, \epsilon\, [0,T]$ and $\|\,.\,\| = \max_{t\in[0,T]} |.|$, suppose that A $= (A_{ij}), B = (B_{ij})$ are non-negative square matrices of order (n), we defined non empty sets as:

$$\left.\begin{aligned} D_F &= x - \frac{Q_1 M_1 \delta_1 T}{\lambda_1} \\ D_H &= y - \frac{Q_2 M_2 \delta_2 T}{\lambda_2} \end{aligned}\right\} \quad (8)$$

Where $x \in D \subseteq R^n, , y \in D_1 \subseteq R^n, T \in R, Q_1, M_1, \delta_1\,, Q_2, M_2, \delta_2$are positive constants .

As well as, we suppose the maximum value of the following matrix:

$$\Lambda_0 = \begin{pmatrix} \frac{Q_1 M_1 \delta_1 T}{\lambda_1} & \frac{Q_1 M_1 \delta_1 T}{\lambda_1} \\ \frac{Q_2 M_2 \delta_2 T}{\lambda_2} & \frac{Q_2 M_2 \delta_2 T}{\lambda_2} \end{pmatrix}$$, less than one i.e.
.

$$\lambda_{max}(\Lambda_0) = \frac{Q_1 M_1 \delta_1 T}{\lambda_1} + \frac{Q_2 M_2 \delta_2 T}{\lambda_2} < 1 \quad (9)$$

Define a sequence of functions $\{x_m(t,x_0)\}_{m=0}^{\infty}$, $\{y_m(t,y_0)\}_{m=0}^{\infty}$ as:

$$\left.\begin{aligned} x_{m+1}(t,x_0) &= x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t,s) F(t,s,x_m(s),y_m(s)) ds ds \\ y_{m+1}(t,y_0) &= y_0 e^{Bt} + \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t,s) H(t,s,x_m(s),y_m(s)) ds ds \end{aligned}\right\} \quad (10)$$

With $x(0) = x_0 e^{At}$, $y(0) = y_0 e^{Bt}$, m=0,1,2…

2. PRELIMINARIES

Definition 2.1 (Syed Abbas, 2011). Assume that $f(x,y)$ is defined on the set (a,b)x$G, G \subset R, f(x,y)$ is said to satisfy Lipschitz condition with respect to the second variable, if for all $x \in (a,b)$ and for any $y_1, y_2 \in G$

$$|f(x,y_1) - f(x,y_2)| \le \xi |y_1 - y_2|$$

where $\xi > 0$ does not depend on $x \in (a,b)$.

Theorem 2.1: (Ascoli-Arzela theorem). Let Ω be a compact Hausdorff metric space. Then $M \subset C(\Omega)$ is relatively compact if and only if M is uniformly bounded and uniformly equicontinuous.

Theorem 2.2: (Krasnoselskii fixed point theorem). Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) Ax + By ∈ M whenever x, y ∈ M (ii) A is compact and continuous (iii) B is a contraction mapping. Then there exists z ∈ M such that z = Az + Bz.

Theorem 2.3 : (Raad N. Butri*s*, Hewa S. Fari*s*, 2015). Let $\{f\}_{n=1}^{\infty}$ be a sequence of real value function on the set E then $\{f\}_{n=1}^{\infty}$ is uniformly convergent on E if and only if given $\in > 0$ then there exist $N \epsilon Z^+$ such that :

$$|f_m(x) - f_n(x)| < \in \qquad (m, n \ge N\,, x \epsilon \text{E})$$

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Theorem 2.4: (Banach's Fixed Point theorem). Let (X, d) be a complete metric space and let $T: X \to X$ be a contraction mapping. Then T has a unique fixed-point x∗ and for any $x \in X$ the sequence $\{T^n(x)\}_{n=1}^{\infty}$ converges to x∗.

Theorem 2.5: (Azhar H.Sallo , 2006). If the function $f(x, y)$ satisfy the existence and uniqueness theorem for IVP(1), then the successive approximation $y_n(x)$ converges to the unique solution $y(x)$ of the IVP(1):

$$\frac{dy}{dx} = f(x, y) \quad , y(x_0) = y_0 \qquad \text{IVP (1)}$$

3. MAIN RESULTS: EXISTENCE

Theorem 3.1: Let the right side of system (1) are defined and continuous on domain (2). Suppose that the vector functions $F(t, s, x(t), y(t))$, $H(t, s, x(t), y(t))$ are satisfying the inequalities (3)-(5) and the conditions (6)-(9).Then there exist a sequences of functions (10) converges uniformly as $m \to \infty$ on domain (2) to the limit functions which satisfying the following integral equations:

$$\left.\begin{aligned} x(t, x_0) &= x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F(t, s, x(s), y(s)) ds ds \\ y(t, x_0) &= y_0 e^{Bt} + \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t, s) H(t, s, x(s), y(s)) ds ds \end{aligned}\right\} \quad (11)$$

Provided that:

$$\|x_\infty(t, x_0) - x_0\| \le \frac{Q_1 M_1 \delta_1 T}{\lambda_1}$$

$$\|y_\infty(t, x_0) - y_0\| \le \frac{Q_2 M_2 \delta_2 T}{\lambda_2}$$

$$\begin{pmatrix} \|x_{m+1}(t, x_0) - x_m(t, x_0)\| \\ \|y_{m+1}(t, y_0) - y_m(t, y_0)\| \end{pmatrix} \le \Lambda_0^m (I - \Lambda_0)^{-1} \varphi_0$$

for all $m \ge 1$, $t \in R^1$

Proof:

By using the sequence of function (10) when m=0, we get:

$$\|x_1(t, x_0) - x_0\| = \left\| x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F(t, s, x_0(s), y_0(s)) ds ds - x_0 e^{At} \right\|$$

$$\le \int_0^t \int_{-\infty}^t \left\| e^{A(t-s)} K(t, s) F(t, s, x_0(s), y_0(s)) ds ds \right\|$$

$$\le Q_1 M_1 \delta_1 \int_0^t \int_{-\infty}^t e^{-\lambda_1(t-s)} ds ds$$

$$\le \frac{Q_1 M_1 \delta_1 T}{\lambda_1}$$

And by the same we have

$$\|y_1(t, y_0) - y_0\| \le \frac{Q_2 M_2 \delta_2 T}{\lambda_2}$$

That is:

$x_1(t, x_0) \epsilon D$ $y_1(t, y_0) \epsilon D_1$, for all $t \epsilon [0, T], x_0 \epsilon D_F$, $y_0 \in D_H$

Suppose that

$x_p(t, x_0) \epsilon D$, $y_p(t, y_0) \epsilon D_1$ for each $x_0 \epsilon D_F$, $y_0 \epsilon D_H$, $p \epsilon Z^+$, t∈ $[0, T]$, by mathematical induction we conclude that:

$x_m(t, x_0) \epsilon D$, $y_m(t, y_0) \epsilon D_1$ for each $x_0 \epsilon D_F$, $y_0 \epsilon D_H$, $m = 0,1,2, \ldots$, t∈ $[0, T]$

To prove that the sequences (10) convergence uniformly in domain (2):

$$\|x_2(t, x_0) - x_1(t, x_0)\| = \Big\| x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F(t, s, x_1(s), y_1(s)) ds ds - x_0 e^{At} \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F(t, s, x_0(s), y_0(s)) ds ds \Big\|$$

$$\le \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) \big\| F(t, s, x_1(s), y_1(s)) - F(t, s, x_0(s), y_0(s)) \big\| ds ds$$

$$\le Q_1 L_1 \delta_1 \int_0^t \int_{-\infty}^t e^{-\lambda(t-s)} \|x_1 - x_0\| + \|y_1 - y_0\| ds ds$$

$$\le \frac{Q_1 L_1 \delta_1 t}{\lambda_1} (\|x_1 - x_0\| + \|y_1 - y_0\|)$$

And by the same

$$\|y_2(t, y_0) - y_1(t, y_0)\| \le \frac{Q_2 L_2 \delta_2 t}{\lambda_2} (\|x_1 - x_0\| + \|y_1 - y_0\|)$$

By the mathematical induction the following inequalities hold:

$$\left.\begin{aligned} &\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le \frac{Q_1L_1\delta_1 t}{\lambda_1}(\|x_m-x_{m-1}\|+\|y_m-y_{m-1}\|) \\ &\|y_{m+1}(t,y_0)-y_m(t,y_0)\| \le \frac{Q_2L_2\delta_2 t}{\lambda_2}(\|x_m-x_{m-1}\|+\|y_m-y_{m-1}\|) \end{aligned}\right\} \quad (12)$$

Rewrite (12) with vector form:

$$\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \\ \|y_{m+1}(t,y_0)-y_m(t,y_0)\|\end{pmatrix} \le \begin{pmatrix}\frac{Q_1L_1\delta_1 t}{\lambda_1} & \frac{Q_1L_1\delta_1 t}{\lambda_1} \\ \frac{Q_2L_2\delta_2 t}{\lambda_2} & \frac{Q_2L_2\delta_2 t}{\lambda_2}\end{pmatrix}\begin{pmatrix}\|x_m-x_{m-1}\| \\ \|y_m-y_{m-1}\|\end{pmatrix}$$

That is:

$$\varphi_{m+1}(t,x_0,y_0) \le \Lambda(t)\varphi_m(t,x_0,y_0) \quad (13)$$

Where:

$$\Lambda(t)=\begin{pmatrix}\frac{Q_1L_1\delta_1 t}{\lambda_1} & \frac{Q_1L_1\delta_1 t}{\lambda_1} \\ \frac{Q_2L_2\delta_2 t}{\lambda_2} & \frac{Q_2L_2\delta_2 t}{\lambda_2}\end{pmatrix}$$

$$,\varphi_{m+1}=\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \\ \|y_{m+1}(t,y_0)-y_m(t,y_0)\|\end{pmatrix}$$

$$,\varphi_m=\begin{pmatrix}\|x_m(t,x_0)-x_{m-1}(t,x_0)\| \\ \|y_m(t,y_0)-y_{m-1}(t,y_0)\|\end{pmatrix}$$

Take the maximum value for both sides of (13):

$$\varphi_{m+1} \le \Lambda_0\,\varphi_m \quad (14)$$

where

$$\Lambda_0=\max_{t\epsilon[0,T]}\Lambda(t)$$

By repletion of (14) we obtain:

$$\varphi_{m+1} \le \Lambda_0^m\,\varphi_0$$

$$\varphi_0 \le \begin{pmatrix}\frac{Q_1M_1\delta_1 T}{\lambda_1} \\ \frac{Q_2M_2\delta_2 T}{\lambda_2}\end{pmatrix}, \quad \sum_{i=1}^m\varphi_i \le \sum_{i=1}^m\Lambda_0^{i-1}\varphi_0 \quad (15)$$

Since the matrix Λ_0 has eigenvalue $\lambda_1=0$,

$$\lambda_2=\lambda_{max}(\Lambda_0)=\frac{Q_1M_1\delta_1 T}{\lambda_1}+\frac{Q_2M_2\delta_2 T}{\lambda_2}<1,$$

the series (15) is uniformly convergent, i.e.

$$\lim_{m\to\infty}\sum_{i=1}^m\Lambda_0^{i-1}\varphi_0=\sum_{i=1}^\infty\Lambda_0^{i-1}\varphi_0 = (I-\Lambda_0)^{-1}\varphi_0 \quad (16)$$

Thus the limiting relation (16) signifies uniform convergence of sequences:

$\{x_m(t,x_0)\}_{m=0}^\infty$, $\{y_m(t,y_0)\}_{m=0}^\infty$, that is: $\lim_{m\to\infty}x_m(t,x_0) = x(t,x_0)$, and $\lim_{m\to\infty}y_m(t,y_0)=y(t,y_0)$

By all conditions and inequalities of the theorem the estimate

$$\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \\ \|y_{m+1}(t,y_0)-y_m(t,y_0)\|\end{pmatrix} \le \Lambda_0^m(I-\Lambda_0)^{-1}\varphi_0$$

Is hold for all m = 0,1,2, …

To prove that $x(t,x_0)\epsilon D$ and $y(t,y_0)\epsilon D_1$ we prove that:

$$\lim_{m\to\infty}\left(x_0e^{At}+\int_0^t\int_{-\infty}^t e^{A(t-s)}K(t,s)F(t,s,x_m(s),y_m(s))dsds\right) = x_0e^{At}+\int_0^t\int_{-\infty}^t e^{A(t-s)}K(t,s)F(t,s,x(s),y(s))dsds \quad (17)$$

$$\lim_{m\to\infty}\left(y_0e^{Bt}+\int_0^t\int_{-\infty}^t e^{B(t-s)}G(t,s)H(t,s,x_m(s),y_m(s))dsds\right) = y_0e^{Bt}+\int_0^t\int_{-\infty}^t e^{B(t-s)}G(t,s)H(t,s,x(s),y(s))dsds \quad (18)$$

We have:

$$\left\|x_0e^{At}+\int_0^t\int_{-\infty}^t e^{A(t-s)}K(t,s)F(t,s,x_m(s),y_m(s))dsds - x_0e^{At} - \int_0^t\int_{-\infty}^t e^{A(t-s)}K(t,s)F(t,s,x(s),y(s))dsds\right\| \quad (19)$$

$$\le \int_0^t\int_{-\infty}^t e^{A(t-s)}K(t,s)\left\|F(t,s,x_m(s),y_m(s))-F(t,s,x(s),y(s))\right\|dsds \le \frac{Q_1L_1\delta_1 T}{\lambda_1}(\|x_m-x\|+\|y_m-y\|)$$

And for the function y(t,y_0) we have

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$$\Big\| y_0 e^{Bt} + \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t,s) H\big(t,s,x_m(s),y_m(s)\big) ds ds - y_0 e^{Bt} - \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t,s) H\big(t,s,x(s),y(s)\big) ds ds \Big\| \leq \frac{Q_2 L_2 \delta_2 T}{\lambda_2} (\|x_m - x\| + \|y_m - y\|)$$

And since the sequences: $\{x_m(t,x_0)\}_{m=0}^{\infty}$, $\{y_m(t,y_0)\}_{m=0}^{\infty}$ uniformly convergence to $x(t,x_0)$, $y(t,y_0)$ respectively on the interval [0,T], that is (17) ,(18) satisfies.

Uniqueness

Theorem (3.2): If all conditions and assumptions of theorem (1.1) satisfied, then the functions $x(t,x_0)$, $y(t,y_0)$ are unique solution for system (1) on domain (2).

Proof: let

$$\begin{pmatrix} \hat{x}(t,x_0) \\ \hat{y}(t,y_0) \end{pmatrix} = \begin{pmatrix} x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t,s) F\big(t,s,\hat{x}(s),\hat{y}(s)\big) ds ds \\ y_0 e^{Bt} + \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t,s) H\big(t,s,\hat{x}(s),\hat{y}(s)\big) ds ds \end{pmatrix}$$

be another solution for system (1) then:

$$\|x(t,x_0) - \hat{x}(t,x_0)\| \leq \int_0^t \int_{-\infty}^t \Big\| e^{A(t-s)} K(t,s) [F\big(t,s,x(s),y(s)\big) - F\big(t,s,\hat{x}(s),\hat{y}(s)\big)] ds ds \Big\|$$

$$\leq \frac{Q_1 L_1 \delta_1 T}{\lambda_1} (\|x - \hat{x}\| + \|y - \hat{y}\|)$$

And

$$\|y(t,y_0) - \hat{y}(t,y_0)\| \leq \int_0^t \int_{-\infty}^t \Big\| e^{B(t-s)} G(t,s) [H\big(t,s,x(s),y(s)\big) - H\big(t,s,\hat{x}(s),\hat{y}(s)\big)] ds ds \Big\|$$

$$\leq \frac{Q_2 L_2 \delta_2 T}{\lambda_2} (\|x - \hat{x}\| + \|y - \hat{y}\|)$$

Rewrite in vector form:

$$\begin{pmatrix} \|x(t,x_0) - \hat{x}(t,x_0)\| \\ \|y(t,y_0) - \hat{y}(t,y_0)\| \end{pmatrix} \leq \Lambda(t) \begin{pmatrix} \|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\| \end{pmatrix}$$

By take the maximum value for both sides of (19) and reputation it we get:

$$\begin{pmatrix} \|x(t,x_0) - \hat{x}(t,x_0)\| \\ \|y(t,y_0) - \hat{y}(t,y_0)\| \end{pmatrix} \leq \Lambda_0^m \begin{pmatrix} \|x(t) - \hat{x}(t)\| \\ \|y(t) - \hat{y}(t)\| \end{pmatrix} \quad (20)$$

From (19) and condition (9) we have $A_0^m \to 0$ when $m \to \infty$ that is:

$$x(t,x_0) = \hat{x}(t,x_0) \text{ and } y(t,y_0) = \hat{y}(t,y_0)$$

Therefor $x(t,x_0)$, $y(t,y_0)$ is a unique solution for system (1).

Stability

Theorem (3.3): Under the hypothesis and conditions of theorem (11) if $\bar{x}(t,x_0), \bar{y}(t,y_0)$ are any other solution of system (1) ,then the solution are stable if satisfies the inequality:

$$\begin{pmatrix} \|x(t,x_0) - \bar{x}(t,x_0)\| \\ \|y(t,y_0) - \bar{y}(t,y_0)\| \end{pmatrix} \leq \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Where:

$$\bar{x}(t,x_0) = x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t,s) F\big(t,s,\bar{x}(s),\bar{y}(s)\big) ds ds$$

$$\bar{y}(t,y_0) = y_0 e^{Bt} + \int_0^t \int_{-\infty}^t e^{B(t-s)} G(t,s) H\big(t,s,\bar{x}(s),\bar{y}(s)\big) ds ds$$

Proof:

$$\|x(t,x_0) - \bar{x}(t,x_0)\| \leq \frac{Q_1 L_1 \delta_1 T}{\lambda_1} (\|x - \bar{x}\| + \|y - \bar{y}\|) \quad (21)$$

$$\|y(t,y_0) - \bar{y}(t,y_0)\| \leq \frac{Q_2 L_2 \delta_2 T}{\lambda_2} (\|x - \bar{x}\| + \|y - \bar{y}\|) \quad (22)$$

Rewrite (21), (22) in victor form we get:

$$\begin{pmatrix} \|x(t,x_0) - \bar{x}(t,x_0)\| \\ \|y(t,y_0) - \bar{y}(t,y_0)\| \end{pmatrix} \leq \Lambda(t) \begin{pmatrix} \|x(t) - \bar{x}(t)\| \\ \|y(t) - \bar{y}(t)\| \end{pmatrix}$$

By condition (9) and for $\epsilon_1, \epsilon_2 \geq 0$ we have:

$$\begin{pmatrix} \|x(t,x_0) - \bar{x}(t,x_0)\| \\ \|y(t,y_0) - \bar{y}(t,y_0)\| \end{pmatrix} \leq \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \quad (23)$$

By the definition of stability, we find that $\bar{x}(t, x_0), \bar{y}(t, y_0)$ is stable solution for system (1)

Krasnoselskii theorem

Theorem (3.4): Suppose that the vector functions $F(t, s, x(t), y(t))$, $H(t, s, x(t), y(t))$ in system (1) are satisfying the inequalities (6), (7)) ,and let $F_1(t, s, x) \leq N_1$, $F_2(t, s, y) \leq N_2$ $\|x_0 e^{At}\| \leq q$, for all $(t, s, x, y) \epsilon [0, T] \times [0, T] \times D \times D_1$,Then system (1.1) has at least one solution in $[0, T]$ provided:

$$q + \frac{TQ_1\delta_1(N_1+N_2)}{\lambda_1} \leq r \tag{24}$$

$$\frac{Q_1 L_{F1}\delta_1 T}{\lambda_1} < 1 \tag{25}$$

Proof: consider $\Psi = \{x \in C([0, T], R): |x| \leq r\}$ denote the collection of all bounded and continuous function from [0, T] to R.

Assume that our function $F(t, s, x, y)$ in system (1) can be written as the sum of tow functions of the following form:

$$F(t, s, x, y) = F_1(t, s, x) + F_2(t, s, y)$$

Where F_1, F_2 are Lipschitz continuous functions with Lipschitz constant L_{F1}, L_{F2} .Define two operators on Ψ as:

$$(\Omega_1 x)(t) = x_0 e^{At} + \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F_1(t, s, x(s)) ds ds$$

$$(\Omega_2 x)(t) = \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) F_2(t, s, y(x)) ds ds$$

$$\|(\Omega_1 x)(t) + (\Omega_2 y)(t)\| \leq \|x_0 e^{At}\| + \int_0^t \int_{-\infty}^t e^{-\lambda_1(t-s)} K(t, s) \|F_1(t, s, x(s)) + F_2(t, s, y(s))\| ds ds$$

$$\leq q + \frac{Q_1 T\delta_1(N_1+N_2)}{\lambda_1} = r$$

Thus for any $x, y \in \Psi$, $(\Omega_1 x) + (\Omega_2 y) \in \Psi$.

$$\|(\Omega_1 x)(t) - (\Omega_1 y)(t)\| \leq \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) \left\| F_1(t, s, x(s)) - F_1(t, s, y(s)) \right\| ds ds$$

$$\leq Q_1 L_{F1}\delta_1 \|x - y\| \int_0^t \int_{-\infty}^t e^{-\lambda_1(t-s)} ds ds$$

$$\leq \frac{Q_1 L_{F1}\delta_1 T}{\lambda_1} \|x - y\|$$

Thus using (25) we conclude that Ω_1 is a contraction mapping.

For $x \in \Psi$ calculating the norm of the function $F = \Omega_1 + \Omega_2$ we have:
moreover for $y \in \Psi$ we obtain:

$$\|(\Omega_2 y)(t)\| \leq \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) \|F_2(t, s, y(s))\| ds ds$$

$$\leq \frac{Q_1 N_2 \delta_1 T}{\lambda_1} \leq r$$

To prove the continuity of Ω_2 ,let us consider a sequence $y_n(t, s, y)$ converging to y, and taking the norm:

$$\|(\Omega_2 y_n)(t) + (\Omega_2 y)(t)\| \leq \int_0^t \int_{-\infty}^t e^{A(t-s)} K(t, s) \|F_2(t, s, y_n(s)) - F_2(t, s, y(s))\| ds ds$$

$$\leq \frac{Q_1 L_{F2}\delta_1 T}{\lambda_1} \|y_n - y\|$$

And hence whenever $y_n \to y$ the function $\Omega_2 y_n \to \Omega_2 y$,this prove the continuity of Ω_2.

For compactness of Ω_2 , let $t_1 \leq t_2 \leq T$ and taking the norm:

$$\|(\Omega_2 y)(t_2) - (\Omega_2 y)(t_1)\| \leq \int_0^{t_2} \int_{-\infty}^{t_2} \left\| e^{A(t_2-s)} K(t_2, s) F_2(t_2, s, y(s)) \right\| ds ds - \int_0^{t_1} \int_{-\infty}^{t_1} \left\| e^{A(t_1-s)} K(t_1, s) F_2(t_1, s, y(s)) \right\| ds ds$$

$$\leq Q_1 N_2 \int_0^{t_2} \int_{-\infty}^{t_2} e^{-\lambda_1(t_2-s)} ds ds - Q_1 N_2 \int_0^{t_1} \int_{-\infty}^{t_1} e^{-\lambda_1(t_1-s)} ds ds$$

$$\leq \frac{Q_1 N_2 \delta_1}{\lambda_1} (t_2 - t_1)$$

The right-hand side of above expression does not depend on y, thus we conclude that Ω_2 is relatively compact and hence Ω_2 is compact by Arzela -Ascoli theorem. Using Krasnoselskii fixed point theorem we obtain that exist $z \in \Psi$ such that:

$$Fz = F_1 z + F_2 z = z$$

Which is a fixed point of F.

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By the same steps we can prove that

$$Hz = H_1 z + H_2 z = z$$

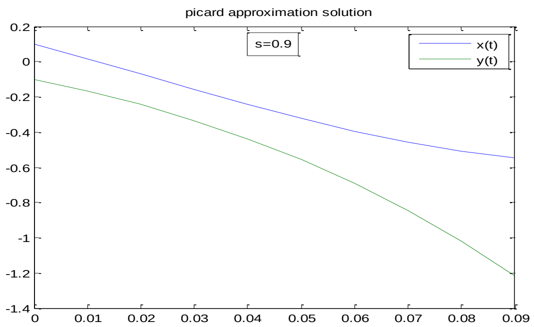
Hence system (1.1) has at least one solution in Ψ .

Example (3.1): consider the following system of integro-differential equations:

$$\left.\begin{aligned} \frac{dx}{dt} &= 0.988x(t) + \int_{-2}^{t} \frac{e^t - e^s}{2} \, \frac{y(s)+\cos(x(s))}{2} \, ds \\ \frac{dy}{dt} &= 1.33y(t) + \int_{-2}^{t} \frac{\cos(t)+\sin(s)}{2} \, \frac{x(s)-\exp(y(s))}{2} \, ds \end{aligned}\right\} \quad (26)$$

For t,s ∈ I = [0,1] here $x_o(t) = 0.1, y_0(t) = 0.1$, A_{1x1}= 0.988 , B_{1x1}= 1.33 , $K(t,s) = \frac{e^t - e^s}{2}$, $G(t,s) = \frac{\cos(t)+\sin(s)}{2}$, $F(t,s,x,y) = \frac{y(s)+\cos(x(s))}{2}$, $H(t,s,x,y) = \frac{x(s)-\exp(y(s))}{2}$, $F_1 = \frac{\cos(x(s))}{2}$, $F_2 = \frac{y(s)}{2}$, $H_1 = \frac{x(s)}{2}$, $H_2 = \frac{-\exp(y(s))}{2}$. Thus, by Theorem (3.1), (3.4) the initial value problem (26) has a unique solution on [0,1], the numerical solution using MATLAB for Picard approximation method are show in figure (1):

	$s = 0$		$s = 0.9$	
t_i	$x(t_i)$	$y(t_i)$	$x(t_i)$	$y(t_i)$
0	0.1	-0.1	0.1	-0.1
0.01	0.0845	-0.1336	0.0160	-0.1667
0.02	0.0685	-0.1715	-0.070	-0.2443
0.03	0.0520	-0.2142	-0.1565	-0.3342
0.04	0.0351	-0.2623	0.2411	-0.4377
0.05	0.0178	-0.3164	-0.3216	-0.5563



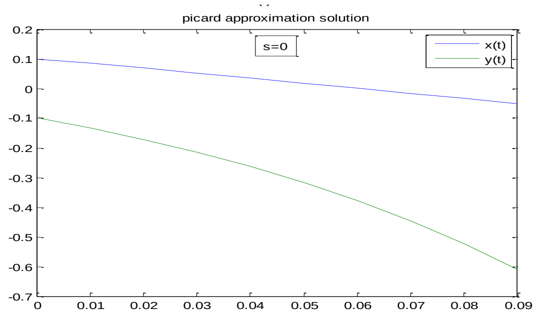


Figure (1)

4. CONCLUSIONS

Based on the results and discussion it can be concluded that: (1) The proof of existence solution for the non-linear system proposed in this paper using the existence and uniqueness theorem needs some hypotheses and conditions, (2). Using the idea of Krasnoselskii fixed point theorem was very effective for the proposed non-linear system of equations. The present work can be extended to boundary value problem.

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