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SOME RESULT IN THE EXISTENCE, UNIQUENESS AND STABILITY SOLUTION OF INTEGRO-DIFFERENTIAL EQUATION

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ABSTRACT

The purpose of this paper is to investigate the existence, uniqueness, and stability solution to a non-linear system of integro-differential equations by using both methods Picard approximation and Banach fixed point theorem. The existence, uniqueness, and stability theorems are established under necessary and sufficient conditions of closed and bounded domain.

Keywords: *Existence, uniqueness, stability solution, Integro-differential equation, Picard approximation method, Banach fixed point theorem.*

1. INTRODUCTION

Differential equations that involve both integrals and derivatives are referred to as integro-differential equations. Existence, uniqueness, and stability solutions of new first-order of integro-differential equations have been extensively studied. Numerous domains, including physics, engineering, and biology, can benefit from the use of these equations (Coddington, and Levinson, 1955; Struble, 1962; Burton, 2005).

The existence and uniqueness of solutions for first-order of integro-differential equations on bounded domains have been studied in several research papers. Guo, Liu, and Zheng et al. studied the existence and uniqueness of solutions for first-order integro-differential equations in Banach using the upper and lower solutions method and the monotone iterative technique (Guo et al., 2019; Mahdi Monje, and Ahmed, 2019; Butris, and Abdi, 2021).

Further, requirements for the local existence, uniqueness, extendibility, and continuity of solutions to a general integral equation with numerous, variable lags were given by Bownds, Cushing, and Schutte (Chalishajar and Kumar, 2018; Hale,1977).

The solutions of specific Volterra type equations using Krasnosel'skii's fixed point theorem (Maleknejad, and Alizadeh, 2009; Islam, and Raffoul, 2013). Other papers include the study of the existence, uniqueness, and stability solutions of integro-differential equations with retarded argument and symmetric matrices (Butris, 2015; Kumar, & Baleanu, 2020), and the operator equation and bounded solutions of integro-differential equations (Ramesh, & Sathiyaraj, 2013).

Butris, and Hasso (1998) studied various theorems on existence and uniqueness of system non-linear integro-differential equations using both the Banach fixed point theorem and the Picard approximation approach, which were proposed by (Rama, 1981).

$$\frac{dx}{dt} = \big(A + B(t)\big)x(t)f(t, x(t), \int_t^{t+T} g\big(s, x(s)\big)ds \quad (i)$$

In this work, we prove the existence, uniqueness and stability solution for another system of non-linear integro-differential equations.

Consider the following system of integro-differential equations which has the form:

$$\left.\begin{aligned}\frac{dx}{dt} &= \big(A_1 + B_1(t)\big)x + f(t, x(t), y(t), z(t), \mu(t))\\ \frac{dy}{dt} &= \big(A_2 + B_2(t)\big)y + g(t, x(t), y(t), z(t), \mathrm{v}(t))\\ \frac{dz}{dt} &= \big(A_3 + B_3(t)\big)z + h(t, x(t), y(t), z(t), \sigma(t))\end{aligned}\right\} (1)$$

where

$$\mu(t) = \int_{-\infty}^{t} G(t,s)x(s)ds,$$

$$\mathrm{v}(t) = \int_{-\infty}^{t} H(t,s)y(s)ds \text{ and}$$

$$\sigma(t) = \int_{-\infty}^{t} K(t,s)z(s)ds$$

Let the vector functions $f(t,x,y,z,\mu)$, $g(t,x,y,z,\mathrm{v})$ and $h(t,x,y,z,\sigma)$ be defined and continuous on the domains:

$$\left.\begin{aligned}(t,x,y,z,\mu) &\in R^1 \times G_1\\ (t,x,y,z,\mathrm{v}) &\in R^1 \times G_2\\ (t,x,y,z,\sigma) &\in R^1 \times G_3\end{aligned}\right\} \qquad (2)$$

where $G_1 = (-\infty,\infty) \times D \times D_1 \times D_2 \times D_\mu$

$G_2 = (-\infty,\infty) \times D \times D_1 \times D_2 \times D_{\mathrm{v}}$

$G_1 = (-\infty,\infty) \times D \times D_1 \times D_2 \times D_\sigma$

where D, D_1 and D_2 are closed bounded domains subset of R^n. Also D_μ, D_{v} and D_σ are bounded domains subset of R^m.

Assume that the vector functions $f(t,x,y,z,\mu)$, $g(t,x,y,z,\mathrm{v})$ and $h(t,x,y,z,\sigma)$ satisfy the following inequalities:

$$\left.\begin{aligned}\|f(t,x,y,z,\mu)\| &\le M_1\\ \|g(t,x,y,z,\mathrm{v})\| &\le M_2\\ \|h(t,x,y,z,\sigma)\| &\le M_3\end{aligned}\right\} \qquad (3)$$

$$\begin{aligned}\|f(t,x_1,y_1,z_1,\mu_1) - f(t,x_2,y_2,z_2,\mu_2)\| &\le L_1\|x_1-x_2\| + L_2\|y_1-y_2\| \\ &+ L_3\|z_1-z_2\| \\ &+ L_4\|\mu_1-\mu_2\| \qquad (4)\end{aligned}$$

$$\begin{aligned}\|g(t,x_1,y_1,z_1,\mathrm{v}_1) - g(t,x_2,y_2,z_2,\mathrm{v}_2)\| &\le \gamma_1\|x_1-x_2\| + \gamma_2\|y_1-y_2\| \\ &+ \gamma_3\|z_1-z_2\| \\ &+ \gamma_4\|\mathrm{v}_1-\mathrm{v}_2\| \qquad (5)\end{aligned}$$

$$\begin{aligned}\|h(t,x_1,y_1,z_1,\sigma_1) - h(t,x_2,y_2,z_2,\sigma_2)\| &\le \rho_1\|x_1-x_2\| + \rho_2\|y_1-y_2\| \\ &+ \rho_3\|z_1-z_2\| \\ &+ \rho_4\|\sigma_1-\sigma_2\| \qquad (6)\end{aligned}$$

for all $t \in R^1, x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, z, z_1, z_2 \in D_2, \mu, \mu_1, \mu_2 \in D_\mu, \mathrm{v}, \mathrm{v}_1, \mathrm{v}_2 \in D_{\mathrm{v}}$ and $\sigma, \sigma_1, \sigma_2 \in D_\sigma$.where $M_1, M_2, M_3, L_1, L_2, L_3, L_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\rho_1, \rho_2, \rho_3, \rho_4$ are positive constants. Also the singular kernels $G(t,s), H(t,s)$ and $K(t,s)$ satisfy the following conditions:

$$\left.\begin{aligned}\|G(t,s)\| &\le Q_1\\ \|H(t,s)\| &\le Q_2\\ \|K(t,s)\| &\le Q_3\end{aligned}\right\} \qquad (7)$$

where Q_1, Q_2 and Q_3 are positive constants. Also, $A_1 = (A_{1ij}), A_2 = (A_{2ij}), A_3 = (A_{3ij})$, $B_1 = (B_{1ij}), B_2 = (B_{2ij})$and $B_3 = B_{3ij}$ are non-negative matrices and $\|.\| = \max_{t\in[0,T]} |.|$, satisfy the following inequalities:

$$\left.\begin{aligned}\|e^{A_1(t-s)}\| &\le \omega_1\\ \|e^{A_2(t-s)}\| &\le \omega_2\\ \|e^{A_3(t-s)}\| &\le \omega_3\end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned}\|B_1(t)\| &\le \vartheta_1\\ \|B_2(t)\| &\le \vartheta_2\\ \|B_3(t)\| &\le \vartheta_3\end{aligned}\right\} \qquad (9)$$

$$\left.\begin{aligned}\|x_0\| &\le \delta_0\\ \|y_0\| &\le \delta_1\\ \|z_0\| &\le \delta_2\end{aligned}\right\} \qquad (10)$$

where $\omega_1, \omega_2, \omega_3, \vartheta_1, \vartheta_2, \vartheta_3, \delta_0, \delta_1$ and δ_2 are positive constants.

Define non-empty sets as follows:

$$\left.\begin{aligned}D_f &= G_1 - \omega_1 b(\vartheta_1\delta_0 + M_1)\\ D_g &= G_2 - \omega_2 b(\vartheta_2\delta_1 + M_2)\\ D_h &= G_3 - \omega_3 b(\vartheta_3\delta_2 + M_3)\end{aligned}\right\} \qquad (11)$$

Furthermore, assume that the largest Eigenvalue of the matrix

$$\Lambda = \begin{pmatrix} N_1 b & b\omega_1 L_2 & b\omega_1 L_3 \\ b\omega_2\gamma_1 & N_2 b & b\omega_2\gamma_3 \\ b\omega_3\rho_1 & \omega_3\rho_2 b & N_3 b \end{pmatrix} \text{less than one}$$

Since the matrix Λ has eigenvalue

$$\left(\frac{1}{2\Psi_1}\left(-\Psi_2 \pm ({\Psi_2}^2 - 4\Psi_1\Psi_3)^{\frac{1}{2}}\right)\right)^{\frac{1}{3}} < 1 \qquad (12)$$

$\Psi_1 = b[N_1 + N_2 + N_3]$,

$$\begin{aligned}\Psi_2 = b^2[&-N_1N_2 - N_1N_3 - N_2N_3 + \omega_2\omega_3\gamma_3\rho_2 \\ &+ \omega_1\omega_2L_2\gamma_1 + \omega_1\omega_3L_3\rho_1]\end{aligned}$$

$$\begin{aligned}\Psi_3 = b^3[&N_1N_2N_3 - \omega_2\omega_3\gamma_3\rho_2N_1 \\ &- \omega_1\omega_2L_2N_3\gamma_1 \\ &+ \omega_1\omega_2\omega_3L_2\gamma_3\rho_1 \\ &+ \omega_1\omega_2\omega_3L_3\gamma_1\rho_2 \\ &- \omega_1\omega_3L_3N_2\rho_1\end{aligned}$$

where$N_1 = \omega_1(\vartheta_1 + l_1 + l_4Q_1), N_2 = \omega_2(\vartheta_2 + \gamma_2 + \gamma_4Q_2)$ and $N_3 = \omega_3(\vartheta_3 + \rho_3 + \rho_4Q_3)$

Suppose that the sequence of functions $\{x_m(t)\}_{m=0}^{\infty}, \{y_m(t)\}_{m=0}^{\infty}$ and $\{z_m(t)\}_{m=0}^{\infty}$ are defined by the following:

$$\begin{aligned}x_{m+1}(t) &= x_0e^{A_1t} \\ &+ \int_0^t e^{A_1(t-s)}[B_1(s)x_m(s) \\ &+ f(s, x_m(s), y_m(s), z_m(s), \mu_m(s))]ds \qquad (13)\end{aligned}$$

$x(0) = x_0$, $m = 0,1,2, \ldots$

$$y_{m+1}(t) = y_0 e^{A_2 t} + \int_0^t e^{A_2(t-s)}[B_2(s)y_m(s) + g(s, x_m(s), y_m(s), z_m(s), \mathrm{v}_m(s))]ds \quad (14)$$

$y(0) = y_0$, $m = 0,1,2, \ldots$

$$z_{m+1}(t) = z_0 e^{A_3 t} + \int_0^t e^{A_3(t-s)}[B_3(s)z_m(s) + h(s, x_m(s), y_m(s), z_m(s), \sigma_m(s))]ds \quad (15)$$

$z(0) = z_0$, $m = 0,1,2, \ldots$

2. EXISTENCE SOLUTION OF (1)

Existence of a solution for the system (1) has been proved by using Picard approximation method.

Theorem1: Let the vector functions $f(t, x, y, z, \mu)$, $g(t, x, y, z, \mathrm{v})$ and $h(t, x, y, z, \sigma)$ are defined and continuous on the domain (2) and satisfy the inequalities from (3) to (10) and conditions (11) and (12).Then the sequence of functions (13), (14) and (15) convergent uniformly on the domains:-

$$\left.\begin{matrix} (t, x_0) \in R^1 \times D_f \\ (t, y_0) \in R^1 \times D_g \\ (t, z_0) \in R^1 \times D_h \end{matrix}\right\} \quad (16)$$

to the limit functions $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ satisfy the following integral equations:

$$x(t) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}[B_1(s)x(s) + f(s, x(s), y(s), z(s), \int_{-\infty}^s G(s,\tau)x(\tau)d\tau]ds \quad (17)$$

$$y(t) = y_0 e^{A_2 t} + \int_0^t e^{A_2(t-s)}[B_2(s)y(s) + g(s, x(s), y(s), z(s), \int_{-\infty}^s H(s,\tau)y(\tau)d\tau]ds \quad (18)$$

$$z(t) = z_0 e^{A_3 t} + \int_0^t e^{A_3(t-s)}[B_3(s)z(s) + h(s, x(s), y(s), z(s), \int_{-\infty}^s K(s,\tau)z(\tau)d\tau]ds \quad (19)$$

a unique solution of (1), provided that :

$$\begin{pmatrix} \|x_m(t) - x_0(t)\| \\ \|y_m(t) - y_0(t)\| \\ \|z_m(t) - z_0(t)\| \end{pmatrix} \le \begin{pmatrix} \omega_1 b(\vartheta_1\delta_0 + M_1) \\ \omega_2 b(\vartheta_2\delta_1 + M_2) \\ \omega_3 b(\vartheta_3\delta_2 + M_3) \end{pmatrix} \quad (20)$$

$$\begin{pmatrix} \|x(t) - x_m(t)\| \\ \|y(t) - y_m(t)\| \\ \|z(t) - z_m(t)\| \end{pmatrix} \le \Lambda^m (E - \Lambda)^{-1}\alpha_1 \quad (21)$$

where

$$\alpha_1 = \begin{pmatrix} \omega_1 b(\vartheta_1\delta_0 + M_1) \\ \omega_2 b(\vartheta_2\delta_1 + M_2) \\ \omega_3 b(\vartheta_3\delta_2 + M_3) \end{pmatrix}$$

Proof: By mathematical induction the sequences (13), (14) and (15) when $m = 0$, provided that:

$$\left.\begin{matrix} \|x_m(t) - x_0\| \le \omega_1 b(\vartheta_1\delta_0 + M_1) \\ \|y_m(t) - y_0\| \le \omega_2 b(\vartheta_2\delta_1 + M_2) \\ \|z_m(t) - z_0\| \le \omega_3 b(\vartheta_3\delta_2 + M_3) \end{matrix}\right\} \quad (22)$$

i.e. $x_m(t) \in G_1, y_m(t) \in G_2, z_m(t) \in G_3, x_0 \in D_f, y_0 \in D_g$ and $z_0 \in D_h$.

Next, we have to prove that the sequences (13), (14) and (15) convergent uniformly on (2) when $m = 1$, to have

$$\|x_2(t) - x_1(t)\| \le \int_0^t \|e^{A_1(t-s)}\|[\|B_1(s)\|\|x_1(s) - x_0\| + L_1\|x_1(s) - x_0\| + L_2\|y_1(s) - y_0\| + L_3\|z_1(s) - z_0\| + L_4\int_{-\infty}^s \|G(s,\tau)\|\|x_1(\tau) - x_0\|d\tau]ds$$

$$\le \int_0^t \omega_1[\vartheta_1\|x_1(s) - x_0\| + L_1\|x_1(s) - x_0\| + L_2\|y_1(s) - y_0\| + L_3\|z_1(s) - z_0\| + L_4 Q_1\|x_1(s) - x_0\|]ds$$

$$\leq \int_0^t \omega_1[(\vartheta_1 + L_1 + L_4 Q_1)\|x_1(s) - x_0\| + L_2\|y_1(s) - y_0\| + L_3\|z_1(s) - z_0\|]ds$$

$$\leq N_1 t\|x_1(t) - x_0\| + t\omega_1 L_2\|y_1(t) - y_0\| + t\omega_1 L_3\|z_1(t) - z_0\|$$

Also,

$$\|y_2(t) - y_1(t)\| \leq t\omega_2\gamma_1\|x_1(t) - x_0\| + N_2 t\|y_1(t) - y_0\| + t\omega_2\gamma_3\|z_1(t) - z_0\|$$

and

$$\|z_2(t) - z_1(t)\| \leq t\omega_3\rho_1\|x_1(t) - x_0\| + \omega_3\rho_2 t\|y_1(t) - y_0\| + tN_3\|z_1(t) - z_0\|$$

By induction, we get

$$\|x_{m+1}(t) - x_m(t)\| \leq N_1 t\|x_m(t) - x_{m-1}(t)\| + t\omega_1 L_2\|y_m(t) - y_{m-1}(t)\| + t\omega_1 L_3\|z_m(t) - z_{m-1}(t)\| \qquad (23)$$

$$\|y_{m+1}(t) - y_m(t)\| \leq t\omega_2\gamma_1\|x_m(t) - x_{m-1}(t)\| + N_2 t\|y_m(t) - y_{m-1}(t)\| + t\omega_2\gamma_3\|z_m(t) - z_{m-1}(t)\| \qquad (24)$$

$$\|z_{m+1}(t) - z_m(t)\| \leq t\omega_3\rho_1\|x_m(t) - x_{m-1}(t)\| + \omega_3\rho_2 t\|y_m(t) - y_{m-1}(t)\| + tN_3\|z_m(t) - z_{m-1}(t)\| \qquad (25)$$

So, from (23) to (25), we get that

$$\begin{pmatrix} \|x_{m+1}(t) - x_m(t)\| \\ \|y_{m+1}(t) - y_m(t)\| \\ \|z_{m+1}(t) - z_m(t)\| \end{pmatrix} \leq \begin{pmatrix} N_1 t & t\omega_1 L_2 & t\omega_1 L_3 \\ t\omega_2\gamma_1 & N_2 t & t\omega_2\gamma_3 \\ t\omega_3\rho_1 & t\omega_3\rho_2 & tN_3 \end{pmatrix} \begin{pmatrix} \|x_m(t) - x_{m-1}(t)\| \\ \|y_m(t) - y_{m-1}(t)\| \\ \|z_m(t) - z_{m-1}(t)\| \end{pmatrix}$$

$$\Omega_{m+1}(t) \leq \Lambda(t)\Omega_m \qquad (26)$$

$$\Omega_{m+1}(t) = \begin{pmatrix} \|x_{m+1}(t) - x_m(t)\| \\ \|y_{m+1}(t) - y_m(t)\| \\ \|z_{m+1}(t) - z_m(t)\| \end{pmatrix}$$

$$\Omega_m = \begin{pmatrix} \|x_m(t) - x_{m-1}(t)\| \\ \|y_m(t) - y_{m-1}(t)\| \\ \|z_m(t) - z_{m-1}(t)\| \end{pmatrix}$$

and $\Lambda(t) = \begin{pmatrix} N_1 t & t\omega_1 L_2 & t\omega_1 L_3 \\ t\omega_2\gamma_1 & N_2 t & t\omega_2\gamma_3 \\ t\omega_3\rho_1 & \omega_3\rho_2 t & tN_3 \end{pmatrix}$

Now for taking the maximum value of both sides of the inequalities (26) gives

$$\Omega_{m+1}(t) \leq \Lambda(t)\Omega_m \qquad (27)$$

where $\Lambda = \max_{t\in[0,T]} \Lambda(t)$, we obtained that

$$\Lambda = \begin{pmatrix} N_1 b & b\omega_1 L_2 & b\omega_1 L_3 \\ b\omega_2\gamma_1 & N_2 b & b\omega_2\gamma_3 \\ b\omega_3\rho_1 & \omega_3\rho_2 b & bN_3 \end{pmatrix}$$

By repetition (27), we get

$$\Omega_{m+1}(t) \leq \Lambda^m \alpha_1 \qquad (28)$$

So,

$$\sum_{i=1}^{m} \Omega_i \leq \sum_{i=1}^{m} \Lambda^{i-1}\alpha_1 \qquad (29)$$

By using (12), then the series (29) is convergent uniformly that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Omega_i = \sum_{i=1}^{\infty} \Lambda^{i-1}\alpha_1 = (E - \Lambda)^{-1}\alpha_1 \qquad (30)$$

Let $\lim_{m\to\infty} \begin{pmatrix} x_m(t) \\ y_m(t) \\ z_m(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ (31)

Since the sequence of functions (13), (14) and (15) are define and continuous in the domain (2) then the limiting vector functions $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is also defined and continuous on the domain (2), hence the vector functions $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a solution of (1).

Theorem 2. With all hypotheses and conditions of theorem 1. Then $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a unique solution of (1).

Proof: Suppose $\begin{pmatrix} x^*(t) \\ y^*(t) \\ z^*(t) \end{pmatrix}$ be another solution of (1).

That is

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$$x^*(t) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}[B_1(s)x^*(s) + f(s, x^*(s), y^*(s), z^*(s), \mu^*(t)]ds \quad (32)$$

$$y^*(t) = y_0 e^{A_2 t} + \int_0^t e^{A_2(t-s)}[B_2(s)y^*(s) + g(s, x^*(s), y^*(s), z^*(s), \sigma^*(t)]ds \quad (33)$$

and

$$z^*(t) = z_0 e^{A_3 t} + \int_0^t e^{A_3(t-s)}[B_3(s)z^*(s) + h(s, x^*(s), y^*(s), z^*(s), \mathrm{v}^*(t)]ds \quad (34)$$

Now,

$$\|x(t) - x^*(t)\| \le \int_0^t \left\|e^{A_1(t-s)}\right\| [\|B_1(s)\|\|x(s) - x^*(s)\| + L_1\|x(s) - x^*(s)\| + L_2\|y(s) - y^*(s)\| + L_3\|z(s) - z^*(s)\| + L_4 \int_{-\infty}^{s} \|G(s,\tau)\|\|x(\tau) - x^*(\tau)\|d\tau]ds$$

$$\le \int_0^t \omega_1[\vartheta_1\|x(s) - x^*(s)\| + L_1\|x(s) - x^*(s)\| + L_2\|y(s) - y^*(s)\| + L_3\|z(s) - z^*(s)\| + L_4 Q_1\|x(s) - x^*(s)\|]ds$$

$$\le \int_0^t \omega_1[(\vartheta_1 + L_1 + L_4 Q_1)\|x(s) - x^*(s)\| + L_2\|y(s) - y^*(s)\| + L_3\|z(s) - z^*(s)\|]ds$$

$$\|x(t) - x^*(t)\| \le N_1 t\|x(t) - x^*(t)\| + t\omega_1 L_2\|y(t) - y^*(t)\| + t\omega_1 L_3\|z(t) - z^*(t)\| \quad (35)$$

and

$$\|y(t) - y^*(t)\| \le t\omega_2\gamma_1\|x(t) - x^*(t)\| + N_2 t\|y(t) - y^*(t)\| + t\omega_2\gamma_3\|z(t) - z^*(t)\| \quad (36)$$

Similarly,

$$\|z(t) - z^*(t)\| \le t\omega_3\rho_1\|x(t) - x^*(t)\| + \omega_3\rho_2 t\|y(t) - y^*(t)\| + tN_3\|z(t) - z^*(t)\| \quad (37)$$

The inequalities (35), (36) and (37) in the vector form:

$$\begin{pmatrix} \|x(t) - x^*(t)\| \\ \|y(t) - y^*(t)\| \\ \|z(t) - z^*(t)\| \end{pmatrix} \le \begin{pmatrix} N_1 b & b\omega_1 L_2 & b\omega_1 L_3 \\ b\omega_2\gamma_1 & N_2 b & b\omega_2\gamma_3 \\ b\omega_3\rho_1 & \omega_3\rho_2 b & tb \end{pmatrix} \begin{pmatrix} \|x(t) - x^*(t)\| \\ \|y(t) - y^*(t)\| \\ \|z(t) - z^*(t)\| \end{pmatrix} \quad (38)$$

From the condition (12) , the system (1) has a unique solution on the domain (2).

3. STABILITY SOLUTIONS OF (1)

In this section, we investigate the stability solution of the system (1), by using the following theorem:

Theorem 4. If the inequalities (3) to (12) are satisfied and $\begin{pmatrix} x^{**}(t) \\ y^{**}(t) \\ z^{**}(t) \end{pmatrix}$ which is another solution of (1), then the solution $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is stable $\forall\, t \ge 0$.

where

$$x^{**}(t) = x_0^{**} e^{A_1 t} + \int_0^t e^{A_1(t-s)}[B_1(s)x^{**}(s) + f(s, x^{**}(s), y^{**}(s), z^{**}(s), \mu^{**}(s)]ds$$

$$y^{**}(t) = y_0^{**} e^{A_2 t} + \int_0^t e^{A_2(t-s)}[B_2(s)y^{**}(s) + g(s, x^{**}(s), y^{**}(s), z^{**}(s), \sigma^{**}(s)]ds$$

and

$$z^{**}(t) = z_0^{**} e^{A_3 t} + \int_0^t e^{A_3(t-s)}[B_3(s)z^{**}(s) + h(s, x^{**}(s), y^{**}(s), z^{**}(s), \mathrm{v}^{**}(s)]ds$$

Proof:

$$\|x(t) - x^{**}(t)\| \le \|x_0 - x_0^{**}\|\left\|e^{A_1(t)}\right\| + N_1 b\|x(t) - x^{**}(t)\| + b\omega_1 L_2\|y(t) - y^{**}(t)\| + b\omega_1 L_3\|z(t) - z^{**}(t)\|$$

But $\left\|e^{A_1(t)}\right\| \le \beta_1, \beta_1 > 0$ and by the definition of stability for $\|x_0 - x_0^{**}\| \le \aleph_1$ we have

$$\|x(t) - x^{**}(t)\| \le \aleph_1\beta_1 + N_1 b\|x(t) - x^{**}(t)\| + b\omega_1 L_2\|y(t) - y^{**}(t)\| + b\omega_1 L_3\|z(t) - z^{**}(t)\| \quad (39)$$

Also,

$$\|y(t) - y^{**}(t)\| \le \|y_0 - y_0^{**}\|\left\|e^{A_2(t)}\right\| + b\omega_2\gamma_1\|x(t) - x^{**}(t)\| + N_2 b\|y(t) - y^{**}(t)\| + b\omega_2\gamma_3\|z(t) - z^{**}(t)\|$$

In addition, But $\left\|e^{A_2(t)}\right\| \le \beta_2, \beta_2 > 0$ and also by $\|y_0 - y_0^{**}\| \le \aleph_2$ gives

$$\|y(t) - y^{**}(t)\| \le \aleph_2\beta_2 + b\omega_2\gamma_1\|x(t) - x^{**}(t)\| + N_2 b\|y(t) - y^{**}(t)\| + b\omega_2\gamma_3\|z(t) - z^{**}(t)\| \quad (40)$$

$$\|z(t) - z^{**}(t)\| \le \|z_0 - z_0^{**}\|\left\|e^{A_3(t)}\right\| + b\omega_3\rho_1\|x(t) - x^{**}(t)\| + \omega_3\rho_2 b\|y(t) - y^{**}(t)\| + bN_3\|z(t) - z^{**}(t)\|$$

But $\left\|e^{A_3(t)}\right\| \le \beta_3, \beta_3 > 0$ and using the definition of stability for $\|z_0 - z_0^{**}\| \le \aleph_3$ produces:

$$\|z(t) - z^{**}(t)\| \le \aleph_3\beta_3 + b\omega_3\rho_1\|x(t) - x^{**}(t)\| + \omega_3\rho_2 b\|y(t) - y^{**}(t)\| + bN_3\|z(t) - z^{**}(t)\| \quad (41)$$

Rewrite the inequalities $(39), (40)$ and (41) in vector form, gives

$$\begin{pmatrix} \|x(t) - x^{**}(t)\| \\ \|y(t) - y^{**}(t)\| \\ \|z(t) - z^{**}(t)\| \end{pmatrix} \le \begin{pmatrix} \aleph_1\beta_1 \\ \aleph_2\beta_2 \\ \aleph_3\beta_3 \end{pmatrix} + \Lambda \begin{pmatrix} \|x(t) - x^{**}(t)\| \\ \|y(t) - y^{**}(t)\| \\ \|z(t) - z^{**}(t)\| \end{pmatrix}$$

the condition (12) and the definition of stability (Ramma 1981) , yields

$$\left(\begin{pmatrix} \|x(t) - x^{**}(t)\| \\ \|y(t) - y^{**}(t)\| \\ \|z(t) - z^{**}(t)\| \end{pmatrix}\right) \le \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}, \ \epsilon_1, \epsilon_2, \epsilon_3 > 0.$$

Therefore, $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ are stable for all $t \ge 0$, thus the equation (1) is stable on the same interval.

4. ANOTHER RESULTS THE SYSTEM OF THE SOLUTION (1)

We can study the solution of integro-differential equations by using Banach fixed point theorem .

Theorem 3. Assume that the vector functions $f(t, x, y, z, \mu), g(t, x, y, z, \text{v})$ and $h(t, x, y, z, \sigma)$ are defined and continuous on the domain (2) and satisfy all inequalities and conditions of theorem 1 . Then the system (1) has a unique solution on the domains (2).

Proof: Let $(D, \|.\|)$ is a Banach space. Define a mapping T on D by:

$$Tx(t) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}[B_1(s)x(s) + f(s, x(s), y(s), z(s), \mu(s))]ds \quad (42)$$

$$Ty(t) = y_0 e^{A_2 t} + \int_0^t e^{A_2(t-s)}[B_2(s)y(s) + g(s, x(s), y(s), z(s), \sigma(s))]ds \quad (43)$$

and

$$Tz(t) = z_0 e^{A_3 t} + \int_0^t e^{A_3(t-s)}[B_3(s)z(s) + h(s, x(s), y(s), z(s), \text{v}(s))]ds \quad (44)$$

It easy to prove that $Tx(t), Ty(t)$ and $Tz(t) \in D$

Now, let $x(t), y(t), z(t), x^*(t), y^*(t)$ and $z^*(t) \in D$,

Then,

$$\|Tx(t) - Tx^*(t)\| \le N_1 b\|x(t) - x^*(t)\| + b\omega_1 L_2\|y(t) - y^*(t)\| + b\omega_1 L_3\|z(t) - z^*(t)\| \quad (45)$$

Also,

$$\|Ty(t)-Ty^*(t)\| \leq b\omega_2\gamma_1\|x(t)-x^*(t)\| + N_2b\|y(t)-y^*(t)\| + b\omega_2\gamma_3\|z(t)-z^*(t)\| \quad (46)$$

$$\|Tz(t)-Tz^*(t)\| \leq b\omega_3\rho_1\|x(t)-x^*(t)\| + \omega_3\rho_2 b\|y(t)-y^*(t)\| + bN_3\|z(t)-z^*(t)\| \quad (47)$$

Rewrite the inequalities (45), (46) and (47) in vector form:

$$\begin{pmatrix}\|Tx(t)-Tx^*(t)\|\\ \|Ty(t)-Ty^*(t)\|\\ \|Tz(t)-Tz^*(t)\|\end{pmatrix} \leq \begin{pmatrix}N_1b & b\omega_1L_2 & b\omega_1L_3\\ b\omega_2\gamma_1 & N_2b & b\omega_2\gamma_3\\ b\omega_3\rho_1 & \omega_3\rho_2b & bN_3\end{pmatrix}\begin{pmatrix}\|Tx(t)-Tx^*(t)\|\\ \|Ty(t)-Ty^*(t)\|\\ \|Tz(t)-Tz^*(t)\|\end{pmatrix}$$

By using the condition (12), so that $\begin{pmatrix}Tx(t)\\ Ty(t)\\ Tz(t)\end{pmatrix}$ is a contraction mapping such that $\begin{pmatrix}Tx(t)\\ Ty(t)\\ Tz(t)\end{pmatrix} = \begin{pmatrix}x(t)\\ y(t)\\ z(t)\end{pmatrix}$ is a fixed point on D and its a unique solution of the system (1) .

5. Conclusion.

This paper is study the existence, uniqueness, and stability solution to a non-linear system of integro-differential equations by using both method Picard approximation and Banach fixed point theorem. Theorems are established under necessary and sufficient conditions of compact. Spaces.

REMARK. Our solutions are global when we use the Picard approximation method while we get the local solutions by using the Banach fixed point theorem.

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