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SOLUTIONS OF NONLINEAR BOUNDARY SYSTEM WITH COUPLED INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT

This article presented some theorems on a novel non-linear multiple Integro-differential equations of boundary T-system. It has been studied the numerical-analytic method and Banach fixed point theorem for the existence and approximation of the solutions over considered boundary system in compact space. In this work, we demonstrate that the mention methods can be discussed and analyzed for the existence and uniqueness, of a solution for the vector system. The paper satisfies the Hölder condition.

Keywords: *Nonlinear boundary system; couple integral boundary condition; existence and uniqueness solution; numerical-analytic method.*

1. INTRODUCTION

Nonlinear boundary value problems play a crucial role in the study and management of real-world nonlinear systems and the advancement of innovations. These problems arise in several branches of science, engineering, and physics as wave equation in the physical differential equation that the determination of normal modes is often stated as boundary value problems. The theory of boundary value problems together with a set of additional limitations on the boundaries has a very wide collection of various methods. Conventionally, these methods can be divided into several main categories, namely analytic methods, functional analytical methods, numerical methods, and numerical-analytic methods (Samoilenko, 1985; Ronto, 2000).

The method proposed of boundary value that corresponds to the problem of a solutions of ordinary differential equation systems of first order with non-linear sides (Samoilenko, 1985). It should be remembered that the numerical-analytical approach is primarily aimed at investigating the qualitative problems of a solution's.

The numerical-analytic method is fairly universal and can be used for both the study of the problem of life and practical solution building. For the given differential equation, a boundary value problem involves finding a solution for the given nonlinear differential equations subject to multi - boundary conditions, which is a prescription for certain combinations of needed solution values and their derivatives at more than one point.

Recently, many types of Integro and Integral differential equations have been used to approximate the periodic solution of various different differential equations such as Volterra, Fredholm, and mixed Volterra Fredholm (Butris, 2020; Zavizion, 2009: Zill 2013). The method of successive approximation (numerical analytic method) is due to the simplicity and possibilities clear to approximate construction of a solutions of integro-differential equations. This study become more general and detailed than those introduced by (Butris, Faris, 2020).

The several lemmas and theorems from the numerical-analytical method, which has been used to study the solutions of the nonlinear boundary T-system (Butris, Faris 2020).

In this paper, the nonlinear Boundary system considered as follows:

$$\left.\begin{aligned} \frac{dx}{dt} &= \big(A_1 + B_1(t)\big)x + \big(A_2 + B_2(t)\big)y + f\big(t, x(t), y(t), u(t)\big) \\ \frac{dy}{dt} &= \big(C_1 + D_1(t)\big)x + \big(C_2 + D_2(t)\big)y + g\big(t, x(t), y(t), v(t)\big) \\ &\text{with Boundary conditions} \\ &\int_0^T\int_0^T x(t)dt\,dt = d_1 + u(T) \\ &\int_0^T\int_0^T y(t)dt\,dt = d_2 + v(T) \end{aligned}\right\} \quad \dots(1)$$

where, $0 < \tau \le t \le T$, d_1, and d_2 are constants, $x \in D_0, y \in D_1, u \in D_u$, and $v \in D_v$. The domains, D_0 and D_1 are closed and bounded subset of R^n. Also D_v and D_u are bounded domains subset of R^m.

Consider the vector functions $f(t, x, y, u)$ and $g(t, x, y, v)$ on the following domains are defined and continuous:

$$\left.\begin{aligned} (t, x, y, u) &\in R^n \times D_0 \times D_1 \times D_u = (-\infty, \infty) \times R^{2n} \times R^m \\ (t, x, y, v) &\in R^n \times D_0 \times D_1 \times D_v = (-\infty, \infty) \times R^{2n} \times R^m \end{aligned}\right\} \quad \dots(2)$$

where, $D_0: \|x - x_0\| \le r_x$, $D_1: \|y - y_0\| \le r_y$, $D_u: \|u\| \le d_u$ and $D_v: \|v\| \le d_v$. And are continuous vector functions in x, y, u, v.

The boundary system (1) verifies the continuous vector functions $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$, where $z_1(t) = B_1(t)x(t, x_0, y_0) + \big(A_2 + B_2(t)\big)y(t, x_0, y_0) + f\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big)$ and $z_2(t) = \big(C_1 + D_1(t)\big)x(t, x_0, y_0) + D_2(t)y(t, x_0, y_0) + g\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big)$ which are defined as follows:

$$x(t, x_0, y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\Big(z_1(s) - f_\Delta\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big) + \zeta^1(t, x_0, y_0)\Big)ds, \quad \dots(3)$$

as $x(0, x_0, y_0) = x_0$ and $m = 0,1, \dots$, where

$$f_\Delta\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big) = A_1 x_0 + \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)} z_1(s)\,ds, \quad \dots(4)$$

$$\zeta^1(t, x_0, y_0) = \frac{{A_1}^2}{T(e^{A_1 T} - TA_1 - I)}\Big(d_1 - \int_0^T\int_0^T F\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big)dt\,dt + u(T) - \frac{x_0 T(e^{A_1 T} - I)}{A_1}\Big), \quad \dots(5)$$

$det(A_1) \ne 0$, $det(e^{A_1 T} - TA_1 - I) \ne 0$ and

$$F\big(t, x(t), y(t), u(t)\big) = \int_0^t e^{A_1(t-s)}\Big(z_1(s) - f_\Delta\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big)\Big)ds. \quad \dots(6)$$

$$y(t, x_0, y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)}\Big(z_2(s) - g_\Delta\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big) + \zeta^2(t, x_0, y_0)\Big)ds, \quad \dots(7)$$

as $y(0, x_0, y_0) = y_0$ and $m = 0,1, \dots$, where

$$g_\Delta\big(t, x(t), y(t), v(t)\big) = C_2 y_0 + \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(T-s)} z_2(s)\,ds, \quad \dots(8)$$

$$\zeta^2(t, x_0, y_0) = \frac{{C_2}^2}{T(e^{C_2 T} - TC_2 - I)}\Big(d_2 - \int_0^T\int_0^T G\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big)dt\,dt + v(T) - \frac{y_0 T(e^{C_2 T} - I)}{C_2}\Big), \quad \dots(9)$$

$det(C_2) \ne 0$, $det(e^{C_2 T} - TC_2 - I) \ne 0$, and

$$G\big(t, x(t), y(t), v(t)\big) = \int_0^t e^{C_2(t-s)}\Big(z_2(s) - g_\Delta\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big)\Big)ds. \quad \dots(10)$$

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Also we have

$$u(t) = \int_{-\infty}^{t} \int_{a}^{b} K_1(t,s)\psi_1\big(t,s,x(s),y(s),\rho(s)\big)\,dt\,ds$$
$$v(t) = \int_{a}^{b} \int_{-\infty}^{t} K_2(t,s)\psi_2\big(t,s,x(s),y(s),\omega(s)\big)\,ds\,dt, \quad \dots(11)$$

$$\rho(s) = \int_{h_1(s)}^{h_2(s)} \big(x(\tau)-y(\tau)\big)\,d\tau$$
$$\omega(s) = \int_{h_3(s)}^{h_4(s)} \big(x(\tau)-y(\tau)\big)\,d\tau \quad \dots(12)$$

Assume that the following inequalities are satisfied by the vector functions, $f(t,x,y,u)$, $g(t,x,y,v)$, $\psi_1(t,s,x,y,\rho)$ and $\psi_2(t,s,x,y,\omega)$:

$$\|f(t,x,y,u)\| \le \vartheta_1, \|g(t,x,y,v)\| \le \vartheta_2, \quad \dots(13)$$

$$\|\psi_1(t,s,x,y,\rho)\| \le \xi_1, \|\psi_2(t,s,x,y,\rho)\| \le \xi_2, \quad \dots(14)$$

$$\|f(t,x_1,y_1,u_1) - f(t,x_2,y_2,u_2)\| \le \Gamma_1\|x_1 - x_2\|^{\alpha} + \Gamma_2\|y_1 - y_2\|^{\beta} + \Gamma_3\|u_1 - u_2\|^{\gamma}, \quad \dots(15)$$

$$\|g(t,x_1,y_1,v_1) - g(t,x_2,y_2,v_2)\| \le \Sigma_1\|x_1 - x_2\|^{\alpha} + \Sigma_2\|y_1 - y_2\|^{\beta} + \Sigma_3\|v_1 - v_2\|^{\gamma}, \quad \dots(16)$$

$$\|\psi_1(t,s,x_1,y_1,\rho_1) - \psi_1(t,s,x_2,y_2,\rho_2)\| \le h_1\|x_1 - x_2\|^{\alpha} + h_2\|y_1 - y_2\|^{\beta} + h_3\|\rho_1 - \rho_2\|^{\gamma}, \quad \dots(17)$$

$$\|\psi_2(t,s,x_1,y_1,\omega_1) - \psi_2(t,s,x_2,y_2,\omega_2)\| \le l_1\|x_1 - x_2\|^{\alpha} + l_2\|y_1 - y_2\|^{\beta} + l_3\|\omega_1 - \omega_2\|^{\gamma}, \quad \dots(18)$$

for all $x, x_1, x_2 \in D_0$, $y, y_1, y_2 \in D_1$, $u, u_1, u_2 \in D_u$ and $v, v_1, v_2 \in D_v$, where ϑ_1, ϑ_2, $\Gamma_1, \Gamma_2, \Gamma_3$, $\Sigma_1, \Sigma_2, \Sigma_3$, h_1, h_2, h_3 and l_1, l_2, l_3 are positive constants, $t \in [0,T]$, and $0 < \alpha, \beta, \gamma < 1$.

From the boundary system (1), the positive matrices $K_1(t,s)$ and $K_2(t,s)$ are isolated singular kernels i. e.

$$\|K_1(t,s)\| \le \delta_1 e^{-\gamma_1(t-s)}$$
$$\|K_2(t,s)\| \le \delta_2 e^{-\gamma_2(t-s)} \quad \dots(19)$$

where δ_1, δ_2, γ_1 and γ_2 are positive constants. Also $A_1 = (A_{1ij})$, $A_2 = (A_{2ij})$, $B_1 = (B_{1ij})$, $B_2 = (B_{2ij})$, $C_1 = (C_{1ij})$, $C_2 = (C_{2ij})$, $D_1 = (D_{1ij})$ and $D_2 = (D_{2ij})$ are non-negative matrices for $i,j = 1,2,\dots,n$ and $\|.\| = \max_{t\in[0,T]} |.|$.

We define the non-empty sets as follows:

$$D_f = D_0 - r_x = D_0 - \left(\varrho_1(t)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{\gamma_1^2}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right) + \varsigma_1(t)\mathrm{R}_1 tH^*{}_1(t)\right), \quad \dots(20)$$

$$D_g = D_1 - r_y = D_1 - \left(\varrho_2(t)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) + \varsigma_2(t)\mathrm{R}_2 tH^*{}_2(t)\right), \quad \dots(21)$$

$$\left\|e^{A_1(t-s)}\right\| \le \mathrm{R}_1, \left\|e^{C_2(t-s)}\right\| \le \mathrm{R}_2, \quad \dots(22)$$

$$H_1(t) = \|h_2(t) - h_1(t)\|, H_2(t) = \|h_4(t) - h_3(t)\|, \quad \dots(23)$$

$$H^*{}_1(t) = \|B_1(t)\|\|x(t)\| + \|A_2 + B_2(t)\|\|y(t)\| + \vartheta_1, \quad \dots(24)$$

$$H^*{}_2(t) = \|C_1 + D_1(t)\|\|x(t)\| + \|D_2(t)\|\|y(t)\| + \vartheta_2. \quad \dots(25)$$

Consider the sequences $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ of a continuous vector functions are defined as:

$$x_{m+1}(t,x_0,y_0) = x_0e^{A_1t} + \int_0^t e^{A_1(t-s)}\Big(z_{1,m}(s) - f_{\Delta,m}\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),u_m(s)\big) + \zeta_m{}^1(t,x_0,y_0)\Big)\,ds, \quad \dots(26)$$

with $x(0,x_0,y_0) = x_0$ and $m = 0,1,\dots$, where

$$f_{\Delta,m}\big(s,x_m(s),y_m(s),u_m(s)\big) = A_1x_0 + \frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z_{1,m}(s)\,ds, \quad \dots(27)$$

$$\zeta^1{}_m(t,x_0,y_0) = \frac{A_1{}^2}{T(e^{A_1T} - TA_1 - I)}\left(d_1 - \int_0^T\int_0^T F\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),u_m(s)\big)\,dt\,dt + u_m(T) - \frac{x_0T}{A_1}\left(e^{A_1T} - I\right)\right), \quad \dots(28)$$

$$F\big(t,x_m(t),y_m(t),u_m(t)\big) = \int_0^t e^{A_1(t-s)}\Big(z_{1,m}(s) -$$

$f_{\Delta,m}\big(t, x_m(t,x_0,y_0), y_m(t,x_0,y_0), u_m(t)\big)\big)\, ds,$ ... (29)

$$u_m(t,x_0,y_0) = \int_{-\infty}^{t}\int_{a}^{b} K_1(t,s)\psi_1\big(t,s,x_m(s),y_m(s),\rho_m(s)\big)\, dt\, ds, \quad ...(30)$$

$$\rho_m(s) = \int_{h_1(s)}^{h_2(s)}\big(x_m(\tau) - y_m(\tau)\big)\, d\tau. \quad ...(31)$$

Also we obtain that

$$y_{m+1}(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)}\Big(z_{2,m}(s) - g_{\Delta,m}\big(t, x_m(t), y_m(t), v_m(t)\big) + \zeta_m{}^2(t,x_0,y_0)\Big)\, ds, \quad ...(32)$$

with $y(0,x_0,y_0) = y_0$ and $m = 0,1,\dots$, where

$$g_{\Delta,m}\big(t, x_m(t), y_m(t), v_m(t)\big) = C_2 y_0 + \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(T-s)} z_{2,m}(s)\, ds, \quad ...(33)$$

$$\zeta^2{}_m(t,x_0,y_0) = \frac{C_2{}^2}{T(e^{C_2 T} - TC_2 - I)}\Bigg(d_2 - \int_0^T\int_0^T G\big(t, x_m(t,x_0,y_0), y_m(t), v_m(t)\big)\, dt dt + v_m(T) - \frac{y_0 T}{C_2}\big(e^{C_2 T} - I\big)\Bigg), \quad ...(34)$$

$$G\big(t, x_m(t), y_m(t), v_m(t)\big) = \int_0^t e^{C_2(t-s)}\Big(z_{2,m}(s) - g_{\Delta.m}\big(t, x_m(t,x_0,y_0), y_m(t,x_0,y_0), v_m(t)\big)\Big)\, ds, \quad ...(35)$$

$$v_m(t,x_0,y_0) = \int_a^b\int_{-\infty}^{t} K_2(t,s)\psi_2\big(t,s,x_m(s),y_m(s),\omega_m(s)\big)\, ds\, dt, \quad ...(36)$$

$$\omega_m(s) = \int_{h_3(s)}^{h_4(s)}\big(x_m(\tau) - y_m(\tau)\big)\, d\tau. \quad ...(37)$$

Consider the matrix, $\varphi_\gamma(T)$'s highest Eigen-value does not exceed one where, $\varphi_\gamma(T) = \begin{pmatrix}\varphi_1(T) & \varphi_2(T)\\ \varphi_3(T) & \varphi_4(T)\end{pmatrix}$, that is

$$\max_{\gamma}\big(\varphi_\gamma(T)\big) = \frac{(\varphi_1(T)+\varphi_4(T))}{2} + \frac{\sqrt{(\varphi_1(T)+\varphi_4(T))^2 - 4(\varphi_1(T)\varphi_4(T) - \varphi_2(T)\varphi_3(T))}}{2} < 1, \quad ...(38)$$

where,

$$\varphi_1(t) = \mathrm{R}_1 t\Bigg(\|B_1(t)\| + \Gamma_1 + \Big(\Gamma_3 + \frac{\varrho_1(t)}{\mathrm{R}_1 t\varsigma_1(t)}\Big)\Big(\big(h_1 + h_3(H_1(t))^\gamma\big)\Big(\frac{\delta_1}{(\gamma_1)^2}\big(e^{-\gamma_1 a} - e^{-\gamma_1 b}\big)e^{\gamma_1 T}\Big)\Big)^\gamma\Bigg), \quad ...(39-i)$$

$$\varphi_2(t) = \mathrm{R}_1 t\Bigg(\|A_2 + B_2(t)\| + \Gamma_2 + \Big(\Gamma_3 + \frac{\varrho_1(t)}{\mathrm{R}_1 t\varsigma_1(t)}\Big)\Big(\big(h_2 + h_3(H_1(t))^\gamma\big)\Big(\frac{\delta_1}{(\gamma_1)^2}\big(e^{-\gamma_1 a} - e^{-\gamma_1 b}\big)e^{\gamma_1 T}\Big)\Big)^\gamma\Bigg), \quad ...(39-ii)$$

$$\varphi_3(t) = R_2 t\Bigg(\|C_1 + D_1(t)\| + \Sigma_1 + \Big(\Sigma_3 + \frac{\varrho_2(t)}{\mathrm{R}_2 t\varsigma_2(t)}\Big)\Big(\big(l_1 + l_3 H_2^\gamma\big)\Big(\frac{\delta_2(b-a)}{\gamma_2}\Big)\Big)^\gamma\Bigg), \quad ...(39-iii)$$

$$\varphi_4(t) = R_2 t\Bigg(\|D_2(t)\| + \Sigma_2 + \Big(\Sigma_3 + \frac{\varrho_2(t)}{\mathrm{R}_2 t\varsigma_2(t)}\Big)\Big(\big(l_2 + l_3 H_2^\gamma\big)\Big(\frac{\delta_2(b-a)}{\gamma_2}\Big)\Big)^\gamma\Bigg). \quad ...(39-iv)$$

$$\varsigma_1(t) = \frac{e^{\|A_1\|T} - \|A_1\|Te^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - T\|A_1\| - \|I\|}, \varsigma_2(t) = \frac{e^{\|C_2\|T} - \|C_2\|Te^{\|C_2\|t} - \|I\|}{e^{\|C_2\|T} - T\|C_2\| - \|I\|}, \quad ...(40)$$

$$\varrho_1(t) = \frac{\|A_1\|(e^{\|A_1\|t} - \|I\|)}{T(e^{\|A_1\|T} - T\|A_1\| - \|I\|)}, \quad \varrho_2(t) = \frac{\|C_2\|(e^{\|C_2\|t} - \|I\|)}{T(e^{\|C_2\|T} - T\|C_2\| - \|I\|)}, \quad ...(41)$$

$$\mu_1(t) = \frac{e^{\|A_1\|t} - \|I\|}{\|A_1\|}, \mu_2(t) = \frac{e^{\|C_2\|t} - \|I\|}{\|C_2\|}. \quad (42)$$

Definition 1. A function $f: S \to \mathbb{R}$ satisfies a Hölder condition of order α where, $0 < \alpha < 1$, on $[a,b] \in \mathbb{R}$, if there is a constant $K > 0$, so that $\forall\ x, y \in [a,b]$, $|f(x) - f(y)| \le K|x - y|^\alpha$.

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Lemma 1. Suppose that $x_i \in \mathbb{R}$ and $q \in (0, \infty)$, then we received that

If $x_i \geq 0$ and $q \geq 1$, then for $1 \leq i \leq m$, $\sum_{i=1}^{m} x_i^q \leq (\sum_{i=1}^{m} x_i)^q \leq m^{q-1} \sum_{i=1}^{m} x_i^q$.

The reverse holds if $0 < q \leq 1$. Hence for $1 \leq i \leq m$, $(\sum_{i=1}^{m} x_i)^q \leq \sum_{i=1}^{m} x_i^q$.

If $x_i, y_i \in \mathbb{R}$ and $0 < q \leq 1$, then for $1 \leq i \leq m$, $\|x_i - y_i\|^q \leq \|x_i - y_i\|$.

Lemma 2. Consider $f(t, x, y)$ be a continuous vector function on, [0, T]. Then $\left\|\int_0^t \left(f(s, x(s), y(s)) - \frac{1}{T}\int_0^T f(\tau, x(\tau), y(\tau))\, d\tau\right) ds\right\| \leq \alpha(t)M$ holds, where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$ and $M = \max_{t\in[0,T]} \|f(t, x(t), y(t))\|$, $\forall\, t \in [0, T]$.

Lemma 3. The following inequalities hold under conditions (23), (37) and (39) also from Lemma (1):

$\|\rho_m(t) - \rho_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma (H_1(t))^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma (H_1(t))^\gamma$,

$\|\omega_m(t) - \omega_{m-1}(t)\|^\gamma \leq \|x_m(t) - x_{m-1}(t)\|^\gamma (H_2(t))^\gamma + \|y_m(t) - y_{m-1}(t)\|^\gamma (H_2(t))^\gamma$.

Theorem1. Suppose that $u(t)$, $v(t)$, $\psi_1(t, s, x, y, w)$ and $\psi_2(t, s, x, y, v)$ be continuous vector functions in the domain (2) and satisfy the conditions and inequalities (17), (18), (30), (31), (36) and (37) and the relations of (39). Then the following inequalities hold:-

$i)\ \|u_m(t) - u_{m-1}(t)\| \leq \left(h_1 + h_3(H_1(t))^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\|x_m(t) - x_{m-1}(t)\| + \left(h_2 + h_3(H_1(t))^\gamma\right)\left(\frac{\delta_1}{\gamma^2{}_1}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 t}\right)\|y_m(t) - y_{m-1}(t)\|$,

$ii)\|v_m(t) - v_{m-1}(t)\| \leq \left(l_1 + l_3(H_2(t))^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|x_m(t) - x_{m-1}(t)\| + \left(l_2 + l_3(H_2(t))^\gamma\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\|y_m(t) - y_{m-1}(t)\|$,

for all $t \in [0, T]$ and $m = 1,2,3, \dots$.

Remark. For the the definitions and lemmas, see (Butris, Faris 2020).

2. APPROXIMATION OF A SOLUTIONS OF BOUNDARY SYSTEM (1).

The following theorem proposed the approximation of the solution of the boundary system's (1):

Lemma 4. Suppose that the vector functions $f(t, x, y, u)$ and $g(t, x, y, v)$ are defined and continuous on $[0, T]$. Therefore the equations (3) and (7) are a solutions of boundary system (1).

Proof. Rewrite the differential equation $\frac{dx}{dt}$ in the form of $\frac{dv}{dt}$ using the boundary system (1) and the assumption that $x = ve^{A_1 t}$ as follows:

$\frac{dv}{dt} = B_1(t)v + (A_2 + B_2(t))we^{C_2 t}e^{-A_1 t} + e^{-A_1 t}f(t, x, y, u)$.

Take the integral on both sides and put in $x = ve^{A_1 t}$, where $z_1(t) = \Big(B_1(t)x(t, x_0, y_0) + (A_2 + B_2(t))y(t, x_0, y_0) + f\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big)\Big)$

to obtain that

$$x(t, x_0, y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} z_1(s)\, ds.$$

Next, we have to find the periodic of $x(t, x_0, y_0)$ and put the periodic solution in boundary condition to have:

$$\Delta = \frac{A_1{}^2}{T(e^{A_1 T} - TA_1 - I)}\left(\frac{x_0 T}{A_1}(e^{A_1 T} - I) + \int_0^T \int_0^T \int_0^t e^{A_1(t-s)} z_1(s)\, ds\, dt\, dt - d_1 - u(T)\right).$$

The solution of, $\frac{dx}{dt}$ in (1) will be:

$$x(t, x_0, y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\left(z_1(s) - f_\Delta\big(t, x(t), y(t), u(t)\big) + \zeta^1(t, x_0, y_0)\right) ds.$$

By the same method where $y = we^{C_2 t}$ and $z_2(t) = \big(C_1 + D_1(t)\big)x(t, x_0, y_0) + D_2(t)y(t, x_0, y_0) + g\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big)$ we get the equation (7) as:

$$y(t, x_0, y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \Big(z_2(s) - g_\Delta\big(t, x(t), y(t), v(t)\big) + \zeta^2(t, x_0, y_0)\Big)\, ds.$$

Lemma 5. Suppose that the vector functions $f(t, x, y, u)$ and $g(t, x, y, v)$ are defined and continuous on $[0, T]$. Then from the inequalities (40) and (41) we see that the vector

$$\begin{pmatrix} \|E_1(t, x_0, y_0)\| \\ \|E_2(t, x_0, y_0)\| \end{pmatrix} \le \begin{pmatrix} \varrho_1(t)\left(d_1 - x_0 T^2 + \frac{\delta_1 \xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ + \varsigma_1(t) \mathrm{R}_1 t H^*{}_1(t) \\ \varrho_2(t)\left(d_2 - y_0 T^2 + \frac{\delta_2 \xi_2}{\gamma_2}(b - a)\right) \\ + \varsigma_2(t) \mathrm{R}_2 t H^*{}_2(t) \end{pmatrix},$$

holds, where the equations (3) and (7) have derived to get the following:

$$E_1(t, x_0, y_0) = \frac{(e^{A_1 t} - I)(A_1 d_1 - x_0 T^2 A_1)}{T(e^{A_1 T} - T A_1 - I)} + \int_0^t e^{A_1(t-s)} \Bigg(z_1(s) - \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\big(z_1(s)\big)\, ds - \frac{A_1{}^2}{T(e^{A_1 T} - T A_1 - I)}\Big(\int_0^T \int_0^T \int_0^t e^{A_1(t-s)} \Big(z_1(s) - \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\big(z_1(s)\big)\, ds\Big)\, ds\, dt\, dt - \int_{-\infty}^T \int_a^b K_1(T, s)\psi_1\big(T, s, x(s), y(s), \rho(s)\big)\, dT\, ds\Big)\Bigg)\, ds,$$
... (43)

$$E_2(t, x_0, y_0) = \frac{(e^{C_2 t} - I)(C_2 d_2 - y_0 T^2 c_2)}{T(e^{C_2 T} - T C_2 - I)} + \int_0^t e^{C_2(t-s)} \Bigg(z_2(s) - \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(t-s)}\big(z_2(s)\big)\, ds - \frac{C_2{}^2}{T(e^{C_2 T} - T C_2 - I)}\Big(\int_0^T \int_0^T \int_0^t e^{C_2(t-s)} \Big(z_2(s) - \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(t-s)}\big(z_2(s)\big)\, ds\Big)\, ds\, dt\, dt - \int_a^b \int_{-\infty}^T K_2(T, s)\psi_2\big(T, s, x(s), y(s), \omega(s)\big)\, ds\, dT\Big)\Bigg)\, ds,$$
... (44)

for $0 \le t \le T$.

Proof. According to (3)-(6), (11)-(12) and (43) also by the inequalities (13), (40) and (41) with condition (24) we have:

$$\|E_1(t, x_0, y_0)\| \le \frac{(e^{\|A_1\| t} - \|I\|)(\|A_1\| d_1 - x_0 T^2 \|A_1\|)}{T(e^{\|A_1\| T} - T\|A_1\| - \|I\|)} + \Bigg\|\int_0^t e^{A_1(t-s)} \Bigg(z_1(s) - \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\big(z_1(s)\big)\, ds - \frac{A_1{}^2}{T(e^{A_1 T} - T A_1 - I)}\Big(\int_0^T \int_0^T \int_0^t e^{A_1(t-s)} \Big(z_1(s) - \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)}\big(z_1(s)\big)\, ds\Big)\, ds\, dt\, dt - \int_{-\infty}^T \int_a^b K_1(T, s)\psi_1\big(T, s, x(s), y(s), \rho(s)\big)\, dT\, ds\Big)\Bigg)\, ds\Bigg\|,$$

$$\le \frac{\|A_1\|(e^{\|A_1\| t} - \|I\|)(d_1 - x_0 T^2)}{T(e^{\|A_1\| T} - T\|A_1\| - \|I\|)} + \frac{e^{\|A_1\| T} - \|A_1\| T e^{\|A_1\| t} - \|I\|}{e^{\|A_1\| T} - T\|A_1\| - \|I\|} \mathrm{R}_1 t H^*{}_1(t) + \frac{\|A_1\|(e^{\|A_1\| t} - \|I\|)}{T(e^{\|A_1\| T} - T\|A_1\| - \|I\|)} \frac{\delta_1 \xi_1}{\gamma_1{}^2}\big(e^{-\gamma_1 a} - e^{-\gamma_1 b}\big)e^{\gamma_1 T},$$

$$\le \frac{\|A_1\|(e^{\|A_1\| t} - \|I\|)\Big(d_1 - x_0 T^2 + \frac{\delta_1 \xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\Big)}{T(e^{\|A_1\| T} - T\|A_1\| - \|I\|)} + \frac{e^{\|A_1\| T} - \|A_1\| T e^{\|A_1\| t} - \|I\|}{e^{\|A_1\| T} - T\|A_1\| - \|I\|} \mathrm{R}_1 t H^*{}_1(t).$$

Thus we obtain that

$$\|E_1(t, x_0, y_0)\| \le \varrho_1(t)\Big(d_1 - x_0 T^2 + \frac{\delta_1 \xi_1}{\gamma_1{}^2}\big(e^{-\gamma_1 a} - e^{-\gamma_1 b}\big)e^{\gamma_1 T}\Big) + \varsigma_1(t) \mathrm{R}_1 t H^*{}_1(t).$$

By repeating iterations (7)-(12) and also (24), (40), and (41) we obtain that

$$\|E_2(t, x_0, y_0)\| \le \varrho_2(t)\Bigg(d_2 - y_0 T^2 + \frac{\delta_2 \xi_2}{\gamma_2}(b - a)\Bigg) + \varsigma_2(t) \mathrm{R}_2 t H^*{}_2(t).$$

So from $\|E_1(t, x_0, y_0)\|$ and $\|E_2(t, x_0, y_0)\|$ we receive the vector form:

$$\begin{pmatrix}\|E_1(t,x_0,y_0)\| \\ \|E_2(t,x_0,y_0)\|\end{pmatrix} \leq \begin{pmatrix} \varrho_1(t)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ +\varsigma_1(t)\mathrm{R}_1 tH^*{}_1(t) \\ \varrho_2(t)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) \\ +\varsigma_2(t)\mathrm{R}_2 tH^*{}_2(t) \end{pmatrix}$$

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3. EXISTENCE OF A SOLUTIONS OF BOUNDARY SYSTEM (1).

The following theorem proposed the existence of the solution of the boundary system's (1):

Theorem2. Consider $f(t,x,y,u)$ and $g(t,x,y,v)$ be vector functions on the domain (2) which are defined, continuous and satisfy all inequalities (15)-(19), conditions (22)-(23) and (38)-(41). Then the function's sequences (26) and (32) of converge uniformly as $m \to \infty$ to the limit functions $x^o(t,x_0,y_0)$ and $y^o(t,x_0,y_0)$ are a solutions of boundary system (1):

$$\begin{pmatrix}\|x^o(t,x_0,y_0) - x_0(t,x_0,y_0)\| \\ \|y^o(t,x_0,y_0) - y_0(t,x_0,y_0)\|\end{pmatrix} \leq \begin{pmatrix} \varrho_1(t)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ +\varsigma_1(t)\mathrm{R}_1 tH^*{}_1(t) \\ \varrho_2(t)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) \\ +\varsigma_2(t)\mathrm{R}_2 tH^*{}_2(t) \end{pmatrix}, \quad \ldots(45)$$

$$\begin{pmatrix}\|x^o(t,x_0,y_0) - x_m(t,x_0,y_0)\| \\ \|y^o(t,x_0,y_0) - y_m(t,x_0,y_0)\|\end{pmatrix} \leq \varphi_\gamma^m(T)\left(I - \varphi_\gamma(T)\right)^{-1}\Omega_1(T), \quad \ldots(46)$$

where, $\Omega_1(T) = \begin{pmatrix} \varrho_1(t)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ +\varsigma_1(t)\mathrm{R}_1 tH^*{}_1(t) \\ \varrho_2(t)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) \\ +\varsigma_2(t)\mathrm{R}_2 tH^*{}_2(t) \end{pmatrix}$

and I is an identity matrix.

Proof: The function's sequences $\{x_m(t,x_0,y_0)\}_{m=1}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=1}^{\infty}$ are defined on (26) and (32), continuous on the domain (2). Firstly, by lemma (5) and from (26) where, $z_{1,0}(t) = B_1(t)x_0(t,x_0,y_0) + \big(A_2 + B_2(t)\big)y_0(t,x_0,y_0) + f\big(t, x_0(t,x_0,y_0), y_0(t,x_0,y_0), u_0(t)\big)$ we get:

$$x_1(t,x_0,y_0) = x_0e^{A_1t} + \int_0^t e^{A_1(t-s)}\Bigg(z_{1,0}(s) - \Big(A_1x_0 + \frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z_{1,0}(s)\big)ds\Big) + \Bigg(\frac{A_1{}^2}{T(e^{A_1T}-TA_1-I)}\Bigg(d_1 - \int_0^T\int_0^T\Big(\int_0^t e^{A_1(t-s)}\Big(z_{1,0}(s) - \Big(A_1x_0 + \frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z_{1,0}(s)\big)ds\Big)\Big)ds\Big)dt\,dt + u_0(T) - \frac{x_0T}{A_1}(e^{A_1T}-I)\Bigg)\Bigg)\Bigg)ds,$$

$$= x_0e^{A_1t} + \int_0^t e^{A_1(t-s)}\Big(z_{1,0}(s) - A_1x_0 + \frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z_{1,0}(s)\big)ds + \frac{A_1{}^2d_1}{T(e^{A_1T}-TA_1-I)} - \frac{A_1{}^2}{T(e^{A_1T}-TA_1-I)}\int_0^T\int_0^T\int_0^t e^{A_1(t-s)}\big(z_{1,0}(s)\big)ds\,dt\,dt + \frac{A_1{}^2}{T(e^{A_1T}-TA_1-I)}\int_0^T\int_0^T\int_0^t e^{A_1(t-s)}(A_1x_0)\,ds\,dt\,dt - \frac{A_1{}^2}{T(e^{A_1T}-TA_1-I)}\int_0^T\int_0^T\int_0^t e^{A_1(t-s)}\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z_0(s)\big)ds\,ds\,dt\,dt + \frac{A_1{}^2}{T(e^{A_1T}-TA_1-I)}u_0(T) - \frac{x_0A_1(e^{A_1T}-I)}{(e^{A_1T}-TA_1-I)}\Big)ds,$$

$$= x_0 + \frac{A_1d_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)} + \frac{x_0(e^{A_1t}-I)T(e^{A_1T}-A_1T-I)}{T(e^{A_1T}-TA_1-I)} - \frac{x_0(e^{A_1t}-I)(e^{A_1T}-I)}{(e^{A_1T}-TA_1-I)} + \int_0^t e^{A_1(t-s)}\big(z_{1,0}(s)\big)ds - \frac{A_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)}\int_0^T\int_0^T\int_0^t e^{A_1(t-s)}\big(z_{1,0}(s)\big)ds\,dt\,dt + \frac{A_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)}\big(u_{1,0}(T)\big),$$

$$= x_0 + \frac{A_1(e^{A_1t}-I)(d_1-x_0T^2)}{T(e^{A_1T}-TA_1-I)} + \int_0^t e^{A_1(t-s)}\big(z_{1,0}(s)\big)ds - \frac{A_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)}\int_0^T\int_0^T\int_0^t e^{A_1(t-s)}\big(z_{1,0}(s)\big)ds\,dt\,dt + \frac{A_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)}\big(u_{1,0}(T)\big).$$

Then by the mathematical induction and where, $m = 0$ we get the norm

$$\|x_1(t,x_0,y_0) - x_0\| \leq \frac{\|A_1\|(e^{\|A_1\|t} - \|I\|)(d_1 - x_0T^2)}{T(e^{\|A_1\|T} - T\|A_1\| - \|I\|)} +$$

$\left(\frac{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)-\|A_1\|T(e^{\|A_1\|t}-\|I\|)}{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\right)\mathrm{R}_1 t H^*{}_1(t)+$
$\frac{\|A_1\|(e^{\|A_1\|t}-\|I\|)}{T(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T},$

$\leq$
$\frac{\|A_1\|(e^{\|A_1\|t}-\|I\|)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)}{T(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}+$
$\frac{e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-T\|A_1\|-\|I\|}\mathrm{R}_1 t H^*{}_1(t).$

$\|x_1(t,x_0,y_0)-x_0\|\leq \varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)+\varsigma_1(t)\mathrm{R}_1 t H^*{}_1(t).$

Using the same iterations as equation (32) also (24), (40) and (41) we get:

$\|y_1(t,x_0,y_0)-y_0\|\leq \varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)\mathrm{R}_2 t H^*{}_2(t).$

For $m\geq 1$ and by mathematical induction we have obtained that:

$\|x_m(t,x_0,y_0)-x_0\|\leq \varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{\gamma_1{}^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)+\varsigma_1(t)\mathrm{R}_1 t H^*{}_1(t),$

$\|y_m(t,x_0,y_0)-y_0\|\leq \varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)\mathrm{R}_2 t H^*{}_2(t).$

Beside that from (20), (21) and $\forall\ t\in[0,T]$ where $x_0\in D_f$ and $y_0\in D_g$ we obtain that $x_m(t,x_0,y_0)\in D_0$ and $y_m(t,x_0,y_0)\in D_1$.

In addition, we have to demonstrate the sequences $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ are uniformly convergent on (2). Then by lemma (5) and (26)-(31) when $m=1$ we obtain that

$\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|\leq \Big\|\left(I-\frac{A_1T(e^{A_1t}-I)}{(e^{A_1T}-TA_1-I)}\right)\int_0^t e^{A_1(t-s)}\Big[B_1(s)x_1(s)+(A_2+B_2(s))y_1(s)+f(s,x_1(s),y_1(s),u_1(s))-\big(B_1(s)x_0+(A_2+B_2(s))y_0+f(t,x_0,y_0,u_0)\big)\Big]ds\Big\|+\Big\|\frac{A_1(e^{A_1t}-I)}{T(e^{A_1T}-TA_1-I)}[u_1(T)-u_0(T)]\Big\|,$

$\leq \mathrm{R}_1 t\frac{e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-T\|A_1\|-\|I\|}\Bigg(\|B_1(t)\|+\Gamma_1+\left(\Gamma_3+\frac{\|A_1\|(e^{\|A_1\|t}-\|I\|)}{\mathrm{R}_1 tT(e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|)}\right)\left(\left(h_1+h_3(H_1(t))^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\Bigg)\|x_1(t)-x_0\|+$
$\mathrm{R}_1 t\frac{e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-T\|A_1\|-\|I\|}\Bigg(\|A_2+B_2(t)\|+\Gamma_2+\left(\Gamma_3+\frac{\|A_1\|(e^{\|A_1\|t}-\|I\|)}{\mathrm{R}_1 tT(e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|)}\right)\left(\left(h_2+h_3(H_1(t))^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\Bigg)\|y_1(t)-y_0\|.$

Thus from (39)-(41) we receive

$\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|\leq \mathrm{R}_1 t\varsigma_1(t)\Bigg(\|B_1(t)\|+\Gamma_1+\left(\Gamma_3+\frac{\varrho_1(t)}{\mathrm{R}_1 t\varsigma_1(t)}\right)\left(\left(h_1+h_3(H_1(t))^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\Bigg)\|x_1(t)-x_0\|+$
$\mathrm{R}_1 t\varsigma_1(t)\Bigg(\|A_2+B_2(t)\|+\Gamma_2+\left(\Gamma_3+\frac{\varrho_1(t)}{\mathrm{R}_1 t\varsigma_1(t)}\right)\left(\left(h_2+h_3(H_1(t))^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\right)\right)^{\gamma}\Bigg)\|y_1(t)-y_0\|.$

By the same iterations from (32)-(37) and (39)-(41) we have

$\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\|\leq R_2 t\varsigma_2(t)\Bigg(\|C_1+D_1(t)\|+\Sigma_1+\left(\Sigma_3+\frac{\varrho_2(t)}{\mathrm{R}_2 t\varsigma_2(t)}\right)\left((l_1+l_3H_2^{\gamma})\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\Bigg)\|x_1(t)-x_0\|+R_2 t\varsigma_2(t)\Bigg(\|D_2(t)\|+\Sigma_2+\left(\Sigma_3+\frac{\varrho_2(t)}{\mathrm{R}_2 t\varsigma_2(t)}\right)\left((l_2+l_3H_2^{\gamma})\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\Bigg)\|y_1(t)-y_0\|.$

Since for, $m > 1$ and by induction, we demonstrate from (39) a vector form as follows:

$$\begin{pmatrix} \|x_{m+1}(t) - x_m(t)\| \\ \|y_{m+1}(t) - y_m(t)\| \end{pmatrix} \leq \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_3(t) & \varphi_4(t) \end{pmatrix} \begin{pmatrix} \|x_m(t) - x_{m-1}(t)\| \\ \|y_m(t) - y_{m-1}(t)\| \end{pmatrix},$$

where two sides' maximal t values have been ordered by iterated recurrence on (39)

$$\begin{pmatrix} \|x_{m+1}(T) - x_m(T)\| \\ \|y_{m+1}(T) - y_m(T)\| \end{pmatrix} \leq \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}^m$$

$$\begin{pmatrix} \varrho_1(T)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{{\gamma_1}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ +\varsigma_1(T)\mathrm{R}_1 T{H^*}_1(T) \\ \varrho_2(T)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) \\ +\varsigma_2(T)\mathrm{R}_2 T{H^*}_2(T) \end{pmatrix}.$$

... (47)

From (47) and for any $k \geq 0$ we conclude that

$$\begin{pmatrix} \|x_{m+k}(T) - x_m(T)\| \\ \|y_{m+k}(T) - y_m(T)\| \end{pmatrix} \leq \begin{pmatrix} \|x_{m+k}(T) - x_{m+k-1}(T)\| \\ \|y_{m+k}(T) - y_{m+k-1}(T)\| \end{pmatrix} + \cdots + \begin{pmatrix} \|x_{m+1}(T) - x_m(T)\| \\ \|y_{m+1}(T) - y_m(T)\| \end{pmatrix},$$

Rewrite the vector structure as follows:

$$\Omega_{m+k}(\mathrm{T}) \leq \left(I - \varphi_\gamma(T)\right)^{-1} \varphi_\gamma^m(T)\Omega_1(T),$$

where, $\Omega_{\mathrm{m+1}}(t) = \begin{pmatrix} \|x_{m+k}(T) - x_m(T)\| \\ \|y_{m+k}(T) - y_m(T)\| \end{pmatrix}$, $\varphi_\gamma(T) = \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix}$ and $\Omega_1(T) =$

$$\begin{pmatrix} \varrho_1(t)\left(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{{\gamma_1}^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\right) \\ +\varsigma_1(t)\mathrm{R}_1 t{H^*}_1(t) \\ \varrho_2(t)\left(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\right) \\ +\varsigma_2(t)\mathrm{R}_2 t{H^*}_2(t) \end{pmatrix}.$$

As a result of condition (38) we get $\lim_{n\to\infty} \varphi_\gamma^m(t) = 0$. Accordingly, $\{x_m(t, x_0, y_0)\}_{m=0}^\infty$ and $\{y_m(t, x_0, y_0)\}_{m=0}^\infty$ sequences of the function converge uniformly on the domains (20) and (21).

So the function's sequences $\{x_m(t, x_0, y_0)\}_{m=0}^\infty$ and $\{y_m(t, x_0, y_0)\}_{m=0}^\infty$ converge uniformly on the domains (20) and (21).

Suppose that, $\lim_{m\to\infty} x_m\,(t, x_0, y_0) = x\,(t, x_0, y_0)$, and $\lim_{m\to\infty} y_m\,(t, x_0, y_0) = y(t, x_0, y_0)$.

4. UNIQUENESS SOLUTION OF BOUNDARY SYSTEM (1).

The uniqueness solution of boundary system (1) is stated by the following theorem.

Theorem 3. With all conditions and hypotheses of previous theorem (4), the solution of boundary system (1), is a unique of (20) and (21).

Proof. Let $r(t, x_0, y_0)$ and $w(t, x_0, y_0)$ be another a solutions of (1), that is

$$r(t, x_0, y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\Big(Z_{1,r}(s) - f_{\Delta,r}\big(t, r(t, x_0, y_0), w(t, x_0, y_0), u_r(t)\big) + {\zeta^1}_r(t, x_0, y_0)\Big)ds,$$

with $r(0, x_0, y_0) = x_0$, $Z_{1,r}(s) = B_1(s)r(s) + \big(A_2 + B_2(s)\big)w(s) + f\big(s, r(s), w(s), u_r(s)\big)$,

$$w(t, x_0, y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)}\Big(Z_{2,r}(s) - g_{\Delta,w}\big(t, r(t, x_0, y_0), w(t, x_0, y_0), v_r(t)\big) + {\zeta^2}_w(t, x_0, y_0)\Big)ds,$$

with $w(0, x_0, y_0) = y_0$, $m = 0,1,2, \ldots.$

We have

$$\|x(t, x_0, y_0) - r(t, x_0, y_0)\| \leq \mathrm{R}_1 t \frac{e^{\|A_1\|T} - \|A_1\|Te^{\|A_1\|t} - \|I\|}{(e^{\|A_1\|T} - T\|A_1\| - \|I\|)}\Bigg(\|B_1(t)\| + \Gamma_1 + \left(\Gamma_3 + \frac{\|A_1\|(e^{\|A_1\|t} - \|I\|)}{\mathrm{R}_1 tT(e^{\|A_1\|T} - \|A_1\|Te^{\|A_1\|t} - \|I\|)}\right)\Bigg(\Big(h_1 + h_3\big(H_1(t)\big)^\gamma\Big)\left(\frac{\delta_1}{(\gamma_1)^2}\big(e^{-\gamma_1 a} - e^{-\gamma_1 b}\big)e^{\gamma_1 T}\right)\Bigg)^\gamma\Bigg)\|x(t) - r(t)\|$$

$+R_1 t \frac{e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|}{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)} \Bigg(\|A_2+B_2(t)\| + \Gamma_2 + \Bigg(\Gamma_3 + \frac{\|A_1\|(e^{\|A_1\|t}-\|I\|)}{R_1 tT(e^{\|A_1\|T}-\|A_1\|Te^{\|A_1\|t}-\|I\|)} \Bigg) \Bigg(\Big(h_2 + h_3(H_1(t))^{\gamma} \Big) \Big(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b}) e^{\gamma_1 T} \Big) \Bigg)^{\gamma} \Bigg) \|y(t)-w(t)\|$.

$$\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \le \varphi_1(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| + \varphi_2(T)\|y(t,x_0,y_0) - w(t,x_0,y_0)\|. \quad \dots(48)$$

To achieve the norm bellow, the same procedures are followed. Thus

$$\|y(t,x_0,y_0)-w(t,x_0,y_0)\| \le \varphi_3(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| + \varphi_4(T)\|y(t,x_0,y_0) - w(t,x_0,y_0)\|. \quad \dots(49)$$

We receive a vector form, from (48) and (49) as follows:

$$\begin{pmatrix} \|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\| \end{pmatrix} \le \begin{pmatrix} \varphi_1(T) & \varphi_2(T) \\ \varphi_3(T) & \varphi_4(T) \end{pmatrix} \begin{pmatrix} \|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-w(t,x_0,y_0)\| \end{pmatrix}.$$

Hence from condition (38), the greatest Eigen value of, $\varphi_{\gamma}(T)$'s matrix is less than one, thus we deduce that, $x(t)=r(t)$ and $y(t)=w(t)$. This implies that the boundary system (1) has a unique solution.

5. EXISTENCE OF THE FUNCTIONS Δf AND Δg OF BOUNDARY SYSTEM (1).

The existence solution of the boundary system (1) is uniquely linked with the existence of zeros of the functions $\Delta_f(t,x_0,y_0) \in D_{f1}\times D_{g1} \to R$ and $\Delta_g(t,x_0,y_0) \in D_{f2}\times D_{g2} \to R$ defined by (4) and (8) respectively. Therefore the function sequences (26) and (27) are obtained from the approximate solutions (3) and (7).

Theorem 6. Assuming that all of theorem (4)'s assumptions and conditions are fulfilled, the following inequality holds:

$$\begin{pmatrix} \|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\| \\ \|\Delta_g(t,x_0,y_0)-\Delta_{g,m}(t,x_0,y_0)\| \end{pmatrix} \le \mu\varphi_{\gamma}^{m+1}(T)\big(I-\varphi_{\gamma}(T)\big)^{-1}\Omega_1(T), \quad \dots(50)$$

where,

$$\Omega_1(T) = \begin{pmatrix} \varrho_1(T)\Big(d_1 - x_0T^2 + \frac{\delta_1\xi_1}{{\gamma_1}^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 T}\Big) \\ +\varsigma_1(T)R_1TH^*{}_1(T) \\ \varrho_2(T)\Big(d_2 - y_0T^2 + \frac{\delta_2\xi_2}{\gamma_2}(b-a)\Big) \\ +\varsigma_2(T)R_2TH^*{}_2(T) \end{pmatrix}$$

, $\mu=(\mu_1,\mu_2)$, $\mu_1 = \frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|}$ and $\mu_2 = \frac{\|C_2\|}{e^{\|C_2\|T}-\|I\|}$ for all $m \ge 0$.

Proof. From the equations (3) and (22) we have

$$\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\| \le \Bigg\| \frac{A_1}{e^{A_1T}-I}\Big(\frac{e^{A_1T}-I-TA_1e^{A_1t}}{(e^{A_1T}-TA_1-I)}\Big)\int_0^T e^{A_1(T-s)}\big(z_1(s)-z_{1,m}(s)\big)ds + \frac{T{A_1}^2}{(e^{A_1T}-TA_1-I)}\int_0^t e^{A_1(t-s)}\big(z_1(s)-z_{1,m}(s)\big)ds \Bigg\| + \frac{\|A_1\|^2}{T(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\big\|\big(u(T)-u_m(T)\big)\big\|$$

$$\le \Big[\frac{\|A_1\|R_1t}{e^{\|A_1\|T}-\|I\|}\frac{(e^{\|A_1\|T}-T\|A_1\|e^{\|A_1\|t}-\|I\|)}{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)} + \frac{T\|A_1\|^2R_1t}{e^{\|A_1\|T}-T\|A_1\|-\|I\|}\Big]\big\|\big(z_1(s)-z_{1,m}(s)\big)\big\| + \Big[\frac{\|A_1\|R_1T}{e^{\|A_1\|T}-\|I\|}\frac{(e^{\|A_1\|T}-T\|A_1\|e^{\|A_1\|t}-\|I\|)}{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\Big]\big\|\big(z_1(s)-z_{1,m}(s)\big)\big\| - \Big[\frac{\|A_1\|R_1t}{e^{\|A_1\|T}-\|I\|}\frac{(e^{\|A_1\|T}-T\|A_1\|e^{\|A_1\|t}-\|I\|)}{(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\Big]\big\|\big(z_1(s)-z_{1,m}(s)\big)\big\| + \frac{\|A_1\|^2}{T(e^{\|A_1\|T}-T\|A_1\|-\|I\|)}\big\|\big(u(T)-u_m(T)\big)\big\|$$

$\le R_1T\mu_1(t)\Bigg(\|B_1(t)\| + \Gamma_1 + \Bigg(\Gamma_3 + \frac{\varrho_1(t)}{R_1t\varsigma_1(t)}\Bigg)\Bigg(\Big(h_1 + h_3(H_1(t))^{\gamma}\Big)\Big(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a} - e^{-\gamma_1 b})e^{\gamma_1 T}\Big)\Bigg)^{\gamma}\Bigg)\|x_1(t)-x_0\| +$

$R_1T\mu_1(t)\Bigg(\|A_2+B_2(t)\| + \Gamma_2 + \Bigg(\Gamma_3 +$

$\frac{\varrho_1(t)}{R_1 t\varsigma_1(t)}\Big)\left(\left(h_2+h_3\left(H_1(t)\right)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}\left(e^{-\gamma_1 a}-e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right)\right)^{\gamma}\Big)\|y_1(t)-y_0\|$

$$\left\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\right\|\leq \mu_1\varphi_1(t)\|x(t)-x_m(t)\|+\mu_1\varphi_2(t)\|y(t)-y_m(t)\| \quad \ldots(51)$$

We get the same results under the same inequality and constraints and by doing the same steps

$$\left\|\Delta_g(t,x_0,y_0)-\Delta_{g,m}(t,x_0,y_0)\right\|\leq \mu_2\varphi_1(t)\|x(t)-x_m(t)\|+\mu_2\varphi_2(t)\|y(t)-y_m(t)\|. \quad \ldots(52)$$

Rewrite (51) and (52) in a vector form as:

$$\begin{pmatrix}\left\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\right\|\\ \left\|\Delta_g(t,x_0,y_0)-\Delta_{g,m}(t,x_0,y_0)\right\|\end{pmatrix}=\begin{pmatrix}\mu_1\varphi_1(t) & \mu_1\varphi_2(t)\\ \mu_2\varphi_1(t) & \mu_2\varphi_2(t)\end{pmatrix}\begin{pmatrix}\|x(t)-x_m(t)\|\\ \|y(t)-y_m(t)\|\end{pmatrix},$$

$$\begin{pmatrix}\left\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\right\|\\ \left\|\Delta_g(t,x_0,y_0)-\Delta_{g,m}(t,x_0,y_0)\right\|\end{pmatrix}\leq \mu\varphi_{\gamma}^{m+1}(T)\left(I-\varphi_{\gamma}(T)\right)^{-1}\Omega_1(T).$$

Thus, we conclude that from above vector and also the periodic functions $\Delta_f(t,x_0,y_0)$ and $\Delta_g(t,x_0,y_0)$, ∃ an isolated singular points such that $\Delta_f(t,x_0,y_0)=0$ and $\Delta_g(t,x_0,y_0)=0$, i. e. the boundary system (1) has periodic solutions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$.

Theorem 4. Assume that the boundary system (1) is defined on the intervals $a\leq x\leq b$ and $c\leq y\leq d$. Then for $m\geq 1$ the vector function sequences $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_{g,m}(t,x_0,y_0)$, which are defined in (27) and (33) satisfy the following inequalities:

$$\left.\begin{array}{l}\min\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta_{f,m}(t,x_0,y_0)\leq -\omega_{1m}\\ \max\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta_{f,m}(t,x_0,y_0)\geq \omega_{1m}\end{array}\right\}, \quad \ldots(53)$$

$$\left.\begin{array}{l}\min\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c1+e_2\leq y\leq d_1-e_2}}\Delta_{g,m}(t,x_0,y_0)\leq -\omega_{1m}\\ \max\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c1+e_2\leq y\leq d_1-e_2}}\Delta_{g,m}(t,x_0,y_0)\geq \omega_{1m}\end{array}\right\}. \quad \ldots(54)$$

So the boundary system (1) has a solutions, $x=x(t,x_0,y_0)$ and $y=y(t,x_0,y_0)$ such that

$$x_0\in\left[a_1+\varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{{\gamma_1}^2}\left(e^{-\gamma_1 a}-e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right)+\varsigma_1(t)R_1tH^*{}_1(t),\, b_1-\varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{{\gamma_1}^2}\left(e^{-\gamma_1 a}-e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right)+\varsigma_1(t)R_1tH^*{}_1(t)\right] \quad \ldots(55)$$

and

$$y_0\in\left[c_1+\varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)R_2tH^*{}_2(t),\, d_1-\varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)R_2tH^*{}_2(t)\right]. \quad \ldots(56)$$

Proof. Consider the points x_1 and x_2 be defined in interval, $\left[a_1+\varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{{\gamma_1}^2}\left(e^{-\gamma_1 a}-e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right)+\varsigma_1(t)R_1tH^*{}_1(t), b_1-\varrho_1(t)\left(d_1-x_0T^2+\frac{\delta_1\xi_1}{{\gamma_1}^2}\left(e^{-\gamma_1 a}-e^{-\gamma_1 b}\right)e^{\gamma_1 T}\right)+\varsigma_1(t)R_1tH^*{}_1(t)\right]$, also y_1 and y_2 be defined in interval , $\left[c_1+\varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)R_2tH^*{}_2(t), d_1-\varrho_2(t)\left(d_2-y_0T^2+\frac{\delta_2\xi_2}{\gamma_2}(b-a)\right)+\varsigma_2(t)R_2tH^*{}_2(t)\right]$, such that

$$\left.\begin{array}{l}\Delta_{f,m}(t,x_1,y_1)=\min\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta_{f,m}(t,x_0,y_0)\\ \Delta_{f,m}(t,x_1,y_1)=\max\limits_{\substack{a_1+e_1\leq x\leq b_1-e_1\\ c_1+e_2\leq y\leq d_1-e_2}}\Delta_{f,m}(t,x_0,y_0)\end{array}\right\}, \quad \ldots(57)$$

$$\left.\begin{aligned}\Delta_{g,m}(t,x_1,y_1) &= \min_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}} \Delta_{g,m}(t,x_0,y_0)\\ \Delta_{g,m}(t,x_1,y_1) &= \max_{\substack{a_1+e_1\le x\le b_1-e_1\\ c_1+e_2\le y\le d_1-e_2}} \Delta_{g,m}(t,x_0,y_0)\end{aligned}\right\} \quad \ldots(58)$$

From the inequalities of system (50), we obtain that

$$\begin{pmatrix}\Delta_f(t,x_1,y_1)\\ \Delta_f(t,x_1,y_1)\end{pmatrix} = \begin{pmatrix}\Delta_{f,m}(t,x_1,y_1) + \\ \big(\Delta_f(t,x_1,y_1) - \Delta_{f,m}(t,x_1,y_1)\big) < 0\\ \Delta_{f,m}(t,x_1,y_1) + \\ \big(\Delta_f(t,x_1,y_1) - \Delta_{f,m}(t,x_1,y_1)\big) > 0\end{pmatrix}, \quad (59)$$

$$\begin{pmatrix}\Delta_g(t,x_1,y_1)\\ \Delta_g(t,x_1,y_1)\end{pmatrix} = \begin{pmatrix}\Delta_{g,m}(t,x_1,y_1) + \\ \big(\Delta_g(t,x_1,y_1) - \Delta_{g,m}(t,x_1,y_1)\big) < 0\\ \Delta_{g,m}(t,x_1,y_1) + \\ \big(\Delta_g(t,x_1,y_1) - \Delta_{g,m}(t,x_1,y_1)\big) > 0\end{pmatrix}, \quad (60)$$

and from the continuity of the functions $\Delta_f(t,x_0,y_0)$ and $\Delta_g(t,x_0,y_0)$ also the inequalities (59) and (60), $\exists$ an isolated singular points $(x^o,y^o) = (x_0,y_0)$, $x^o \in [x_1,x_2]$ and $y^o \in [y_1,y_2]$ where, $\Delta_f(t,x_0,y_0)$ and $\Delta_g(t,x_0,y_0)$ are equal to zeros, thus the boundary system (1) has a solutions $x = x(t,x_0,y_0)$ and $y = y(t,x_0,y_0)$.

Remark 1. Theorem 4 is proved when x_0 are y_0 are scalar singular points which should be isolated.

6. CONCLUSIONS

We investigate a solutions for non-linear system of boundary value problems by using the numerical analytic method, which was introduced by Samoilenko, These investigations lead us to improving and extending the above method. Also we expand the results obtained by Samoilenko to change the system of non-linear integro- differential equations with initial condition to a system of non-linear integro- differential equations with boundary conditions.

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