

SOME NEW RESULTS OF A FRACTIONAL VON BERTALANFFY MODEL

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ABSTRACT

In this article, we review the deterministic and perturbed Von Bertalanffy model that has been developing to discuss the existence and uniqueness results of the model. The key research issues are highlighted, and the results and interpretations are summarized. We investigate the effects of changing some of the system's control parameters through numerical simulations. We resolve the Von Bertalanffy ordinary differential equation of fractional order. The analytical solution is obtained.

Keywords: Ordinary differential equation, periodic fluctuations, growth model, fractional derivative, existence and uniqueness.

1. INTRODUCTION

In 1838, Von Bertalanffy presented the Von Bertalanffy model as an organism growth model. This model demonstrates how the tumor's volume changes in relation to the surface area and with cell death. This model, in Vaidya's opinion, accurately forecasts the progression of human tumors (Vaidya (1982)). The Von Bertalanffy ordinary differential equation is written as follows:

$$\frac{dv}{dt} = K(L_{\infty} - v) \quad (1)$$

where t is time, v is length (or some other measure of size), K is the growth rate and L_{∞} , termed 'L infinity' in fisheries science, is the asymptotic length at which growth is zero.

The solution, for a given initial condition $v(0) = v_0$, is:

$$v(t) = L_{\infty} - (L_{\infty} - v_0)e^{-Kt} \quad (2)$$

Eq. (1) presents a sigmoidal curve for $v(t)$ and can model human's height and mass, a fish community, a rabbit (see Zwietering (1990); Waliszewski (2003) and references therein), diverse malicious tumors (Laird (1964); Laird (1965)). For processing biological systems and researching ecology, the traditional Von Bertalanffy system is helpful.

Research has been done to convert deterministic growth models to their stochastic counterparts because deterministic models are in fact insufficient to explain the dynamical process of population expansion (Shabana (2019) and references therein). Researchers have expanded the Von Bertalanffy model in numerous investigations (Lee (2020); Garcia (1983); Konig (2019)). On the other hand, the only models that have been extended to their stochastic counterpart to date are the logistic and Gompertz models (Shabana (2019) and references therein); Al-Saffar (2017); Al-Saffar (2020)).

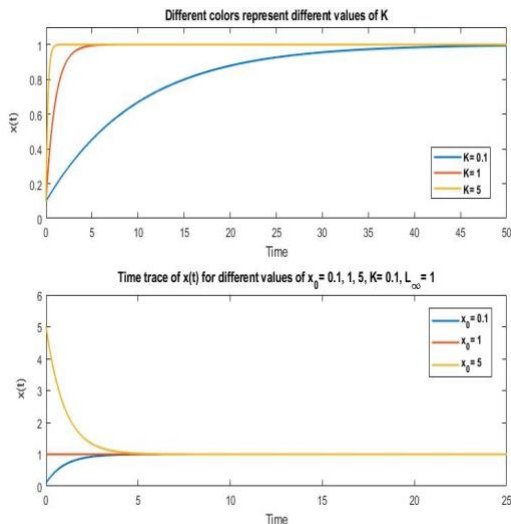
By changing the most important parameters that reflect the presence of noisy behavior, deterministic models can be converted into their stochastic equivalent. The intrinsic growth rate parameter K is perturbed so that the mathematical model in the form of stochastic Von Bertalanffy emerge.

A modulation in the model parameters is examined using fractional derivatives. The fractional operators are increasingly taken into account in biological system modeling due to its interesting advantages in analyzing complex evolution and behaviour patterns linked to memory or fractals. (Wang (2009); Ahmed (2007); Kilicman (2018); Ishak (2022); Ishak

(2023)). There have been many revisions and modifications of the Von Bertalanffy equation (1) discussed, but it is difficult to see a fractional version of this differential equation.

The system in Eq. (1) has one steady state $v^* = L_\infty$. v can represent the population of any species of interest (e.g. tumor, rabbit, bacteria, etc). For a constant $K > 0$, v reaches L_∞ as $t \rightarrow \infty$ regardless of the initial value of $v(t = 0) = v_0$. Finding the numerical and analytical answer to Equation (1) is convenient. One intriguing feature of the Von Bertalanffy equation is that the population will display an equilibrium solution at $v = L_\infty$, as we can see from Fig. (1) which is a constant, unchanging point in time. As a result, the population will stop growing, unlike how it would if K or v were 0. This illustrates that the partial derivative with respect to time is equal to zero for a variety of initial conditions. $\frac{\partial v}{\partial t} = 0$.

For the model in (2), where v_0 is the initial value of v at ($t = 0$). We show the typical time history of $v(t)$ for different values of ω and v_0 in Fig. (1).



For $L_\infty = 1, v_0 = 0.1$, the upper panel from figure (1), it is obvious that the solution show a stable grow at $L_\infty = 1$ for different values of the parameter growth K with only a difference which is the time that the system spent as an initial transient, also, this indicates that for higher values of K , the length (or any other size-measure) of the organism will reach the asymptotic length faster which leads the population to stop expanding early. Whereas in the lower panel where we fix the value of K and change the initial condition, we observe a similar behaviour.

2. VON BERTALANFFY MODEL WITH A PERIODIC MODEULATION

Different growth populations have been frequently modeled using deterministic Von Bertalanffy models. By modulating the growth rate parameter K , the Von Bertalanffy equation with periodic modulation is formed as follows:

$$\frac{dv}{dt} = (\alpha + \beta \cos \omega t) (L_\infty - v) \tag{3}$$

In the case of a periodic fluctuation ($\alpha + \beta \cos(\omega t)$) is included in the growth rate parameter, where α is a constant growth, β and ω are the amplitude and frequency of the modulation, respectively. The analytical solution to Eq. (3) can be found in the following form:

$$v(t) = L_\infty - (L_\infty - v_0)e^{-at - \frac{\beta}{\omega} \sin(\omega t)} \tag{4}$$

where v_0 represents v 's initial value at $t = 0$. The perturbed model is appropriate for providing study of the growth of populations and expansion.

3. PRELIMINARIES

We present these preliminary results in order to explore the presence and uniqueness of the initial value solution for the the following fractional representation of the Von Bertalanffy differential equation has been taken into consideration.

$${}^c D^p v(t) = (\alpha + \beta \cos(\omega t)) (L_\infty - v) \tag{5}$$

with $p \in (0,1)$ and D^p represents the Caputo fractional derivative, with $v(0) = 0$, Where

${}^c D^p$, denotes the Caputo differential operator of order $p, 0 < p < 1, \alpha, \beta, \omega, L_\infty$ are parameters, $t \in [0, T]$. Let's review some fundamental definitions of fractional calculus that will be useful for this study.

Definition 3.1. (Agarwal, (2011)

The Caputo derivative of fractional order p is defined as follows for a continuous function $g: [0, \infty) \rightarrow \mathcal{R}$:

$${}^c D^p(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{n-p-1} g^n(s) ds,$$

$n - 1 < p < n, n = [p] + 1$, where the integer component of the real number p is denoted by $[p]$.

Definition 3.2. (Ishak, (2020)) *The Riemann-Liouville fractional integral of order p is defined by*

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds, \quad p > 0$$

assuming the integral is available.

Theorem 3.1. (Ishak, (2020)).

Assuming that x is a Banach space, Ω is an open bounded subset of x with $\theta \in \Omega$, and $T: \bar{\Omega} \rightarrow x$ is a fully continuous operator, then the following conditions hold:

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega$$

Then T has a fixed point in Ω . \square

Lemma 3.1. (Ahmad (2011))

For $p > 0$, the universal solution of the fractional differential equation is ${}^c D^p v(t) = 0$, is denoted by:

$$v(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$$

where $c_i \in \mathcal{R}, i = 1, 2, \dots, n-1, n = [p] + 1$. \square

The following can be derived from lemma (3.1):

$$I^{p^c} D^p v(t) = v(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathcal{R}, i = 1, 2, \dots, n-1$.

Rewrite the problem (5) as:

$$v(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \varrho(t, v(t)) ds, \quad v(0) = 0$$

where

$$\varrho(t, v(t)) = (\alpha + \beta \cos(\omega t)) (L_\infty - v), \text{ and } \varrho(\alpha, \beta, \omega, t, L_\infty, v(t)) = (\alpha + \beta \cos(\omega t)) (L_\infty - v)$$

4. EXISTENCE RESULTS

Let $\Theta = C([0, T], \mathcal{R})$ stands for the Banach space of all continuous functions from $[0, T] \rightarrow \mathcal{R}$ endowed with the norm defined by:

$$\|v\| = \sup |v(t)|, \quad t \in [0, T]$$

Define an operator $\Pi: \Theta \rightarrow \Theta$ as:

$$(\Pi v)(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \varrho(s, v(s)) ds,$$

Observe that the operator Π must have a fixed point in order for the equation in (5) to have a solution.

Lemma 4.1.

The operator $\Pi: \Theta \rightarrow \Theta$ is completely continuous.

Proof. Let $A \subset \Theta$ be bounded, then $\forall t \in [0, T], v \in A$, there exists a positive constants ξ_1, ξ_2 such that $|v(t)| \leq \xi_1$ and $|\cos(\omega s)| \leq \xi_2$, thus, we have:

$$\begin{aligned} |(\Pi v)(t)| &\leq \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} |\alpha + \beta \cos(\omega t)| (L_\infty - v) ds \\ &\leq \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} (|\alpha L_\infty| - \alpha |v(s)| + \beta L_\infty |\cos(\omega s)| - \beta |\cos(\omega s)| |v(s)|) ds \\ &\leq \frac{\alpha L_\infty - \alpha \xi_1 + \beta L_\infty \xi_2 - \beta \xi_1 \xi_2}{\Gamma(p)} \int_0^t (t-s)^{p-1} ds \\ &\leq \frac{(\alpha L_\infty - \alpha \xi_1 + \beta L_\infty \xi_2 - \beta \xi_1 \xi_2) t^p}{\Gamma(p+1)} \end{aligned}$$

This suggests that $|(\Pi v)(t)|$ is limited, furthermore, for $t_1, t_2 \in [0, T]$, we have

$$\begin{aligned} |(\Pi v)(t_2) - (\Pi v)(t_1)| &\leq \frac{1}{\Gamma(p)} \int_0^{t_2} (t_2 - s)^{p-1} |\varrho(s, v(s))| ds - \frac{1}{\Gamma(p)} \int_0^{t_1} (t_1 - s)^{p-1} |\varrho(s, v(s))| ds \\ &\leq \frac{\alpha L_\infty - \alpha \xi_1 + \beta L_\infty \xi_2 - \beta \xi_1 \xi_2}{\Gamma(p)} \left[\int_0^{t_2} (t_2 - s)^{p-1} ds - \int_0^{t_1} (t_1 - s)^{p-1} ds \right] \\ &\leq \frac{(\alpha L_\infty - \alpha \xi_1 + \beta L_\infty \xi_2 - \beta \xi_1 \xi_2)(t_2^p - t_1^p)}{\Gamma(p+1)} \end{aligned}$$

The Ascoli-Arzela theorem states that $\Pi: \Theta \rightarrow \Theta$ is entirely continuous since the right-hand side of the inequality above goes to zero regardless of $v \in \Theta$ as $t_2 \rightarrow t_1$.

Theorem 4.1.

Let

$$\varrho(\alpha, \beta, \omega, t, L_\infty, v(t)): [0, \infty) \times [0, \infty) \times [0, \infty) \times [0, T] \times [0, \infty) \times \mathcal{R} \rightarrow \mathcal{R}$$

and

$$\lim_{v \rightarrow 0} \frac{\varrho(\alpha, \beta, \omega, t, L_\infty, v(t))}{v} = 0$$

then problem (5) has at least one solution.

Proof. Since

$$\lim_{v \rightarrow 0} \frac{\varrho(\alpha, \beta, \omega, t, L_\infty, v(t))}{v} = 0$$

then there exists a constant $\epsilon > 0$ such that $|\varrho(\alpha, \beta, \omega, t, L_\infty, v(t))| \leq \mu|v|$ for $0 < |v| < \epsilon$ where $\mu > 0$ and

$$\frac{t^p}{\Gamma(p+1)}\mu \leq 1$$

Define $B = v \in \Theta: |v| < \epsilon$ and take $v \in \Theta$ such that $|v| < \epsilon$ that is $v \in \partial B$ by Lemma (4.1), we know that Π is completely continuous and

$$\begin{aligned} & |(\Pi v)(t)| \\ & \leq \frac{1}{\Gamma(p)} \int_0^t (t-s) |\varrho(\alpha, \beta, \omega, s, L_\infty, v(s))| ds \\ & \leq \frac{\mu|v|}{\Gamma(p)} \int_0^t (t-s)^{p-1} ds \\ & \leq \frac{\mu|v|t^p}{\Gamma(p+1)} \leq |v| \end{aligned}$$

Thus we have $|\Pi v(t)| \leq |v|, v \in \partial B$. Since the operator Π has at least one fixed point according to theorem (3.1), it follows that the problem (5) has at least one solution.

Theorem 4.2. Assume that

$$\varrho(\alpha, \beta, \omega, t, L_\infty, v(t)): [0, \infty] \times [0, \infty] \times [0, \infty] \times [0, T] \times [0, \infty] \times \mathcal{R} \rightarrow \mathcal{R}$$

is jointly continuous function satisfying the following condition:

$$\begin{aligned} & \left| \varrho(\alpha, \beta, \omega, t, L_\infty, v(t)) - \varrho(\alpha, \beta, \omega, t, L_\infty, y(t)) \right| \leq L|v - y|, \\ & \forall t \in [0, T], \alpha \in [0, \infty], \beta \in [0, \infty], \omega \in [0, \infty], L_\infty \in [0, \infty], v, y \in \mathcal{R} \end{aligned}$$

With

$$L \leq \frac{0.5\Gamma(p+1)}{t^p}$$

then problem (5) has a unique solution.

Proof. Setting

$$\begin{aligned} & \sup_{t \in [0, T], \alpha \in [0, \infty], \beta \in [0, \infty], \omega \in [0, \infty], L_\infty \in [0, \infty]} \varrho(\alpha, \beta, \omega, t, L_\infty, 0) \\ & = M \end{aligned}$$

and selecting

$$r \geq \frac{2t^p M}{\Gamma(p+1)}$$

we show that $\Pi B_r \subset B_r$ where $B_r = v \in C[0, T]: |v| \leq r$, for $v \in B$, we have:

$$\begin{aligned} & |(\Pi v)(t)| \\ & \leq \max \left[\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} |\varrho(\alpha, \beta, \omega, s, L_\infty, v(s))| ds \right] \\ & \leq \max \left[\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} |\varrho(\alpha, \beta, \omega, s, L_\infty, v(s)) - \varrho(\alpha, \beta, \omega, s, L_\infty, 0) + \varrho(\alpha, \beta, \omega, s, L_\infty, 0)| ds \right] \\ & \leq \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left| \varrho(\alpha, \beta, \omega, s, L_\infty, v(s)) - \varrho(\alpha, \beta, \omega, s, L_\infty, 0) \right| ds \\ & \quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} |\varrho(\alpha, \beta, \omega, s, L_\infty, 0)| ds \\ & \leq \frac{Lrt^p}{\Gamma(p+1)} + \frac{Mt^p}{\Gamma(p+1)} \leq \frac{t^p(Lr+M)}{\Gamma(p+1)} \leq r \end{aligned}$$

For $v, y \in (C[0, 1], \mathcal{R})$ and for $t \in [0, T]$, we obtain:

$$\begin{aligned} & |(\Pi v)(t) - (\Pi y)(t)| \\ & \leq \max \left[\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} |\varrho(\alpha, \beta, \omega, s, L_\infty, v(s)) - \varrho(\alpha, \beta, \omega, s, L_\infty, y(s))| ds \right] \\ & \leq \frac{L|v - y|t^p}{\Gamma(p+1)} \leq B|v - y| \end{aligned}$$

Where

$$B = \frac{Lt^p}{\Gamma(p+1)}$$

As $B < 1$, Π is contraction. Thus, the contraction mapping principle leads to the theorem's conclusion.

5. CONCLUSION

This article's goal is to examine the dynamics of the deterministic, stochastic, and fractional Von Bertalanffy model in order to ascertain how changing parameter values impact population behavior. The impacts of various beginning conditions and K are taken into consideration. After an initial transient, the model continues to behave in the same way throughout time. This indicates that if the length

growth rate of organisms rises, the length (or any other size-measure) of the organism will reach the asymptotic length more quickly, which will cause the population to cease increasing early.

We demonstrated the existence of the solution and its uniqueness using the Banach's fixed point theorem after solving the fractional Von Bertalanffy differential equation with periodic modulation.

The Von Bertalanffy deterministic model has been extended to stochastic setting by perturbing the growth rate parameter K to include $(\alpha + \beta \cos \omega t)$.

It will be interesting to investigate different models using similar methods in the future.

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