CR- Submanifolds of a Nearly Trans-Hyperbolic Sasakian Manifold with a Quarter Symmetric Semi Metric Connection

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Abstract: The object of the present paper is to initiate the study contact CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection. For this, some properties of CR- submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection are investigated which conclude that CR- submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection exists with respect to the $\xi-$horizontal and $\xi-$vertical.

Keyword: CR-submanifolds, nearly trans-hyperbolic Sasakian manifold, quarter symmetric semi metric connection, Gauss and Weingarten equations parallel distributions.

1. Introduction

Bejancu [1] defined the notion of CR-submanifolds of a Kaehler manifold in [2]. After that a number of authors have studied these submanifolds ([10], [13], [14], [18]). Upadhyay and Dube [15] have defined almost contact hyperbolic (f,g,η,ξ) -structure, Dube and Mishra [5] have considered Hypersurfaces immersed in an almost hyperbolic Hermitian manifold also Dube and Niwas [6] worked with almost r-contact hyperbolic structure in a product manifold. Gherghe studied on harmonicity on nearly trans-Sasaki manifolds [7]. Bhatt and Dube [3] studied on CR-submanifolds of trans- hyperbolic Sasakian manifold. Joshi and Dube [9] studied on Semi-invariant submanifold of an almost r-contact hyperbolic metric manifold. Gill and Dube have also worked on CR submanifolds of trans-hyperbolic Sasakian manifolds [8].

Let ∇ be a linear connection in an n-dimensional differentiable manifold $\bar{M}$. The torsion tensor T and the curvature tensor R of ∇ are given respectively by [4]

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [17], S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form. In [11], M. Ahmad, J. B. Jun and A. Haseeb studied some properties of hypersurfaces of an almost *r*-paracontact Riemannian manifold with quarter symmetric semi metric connection. In [12], M. Ahmad, C. Ozgur and A. Haseeb studied properties of hypersurfaces of an almost *r*-paracontact Riemannian manifold with quarter symmetric non-metric connection.

In this paper, CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection are investigated. Parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric semi metric connection are also discussed.

2. Preliminaries

Let $\bar{M}$ be an n dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ,ξ,η,g) where a tensor ϕ of type (1, 1), a vector field ξ, called structure vector field and η, the dual 1-form of is a 1-form ξ satisfying the following

(2.1) $\phi^2 X = X - \eta(X)\xi, \quad g(X,\xi) = \eta(X)$

(2.2) $\phi(\xi) = 0, \quad \eta o \phi = 0, \quad \eta(\xi) = -1$

(2.3) $g(\phi X,\phi Y) = -g(X,Y) - \eta(X)\eta(Y),$

for any X, Y tangents to $\bar{M}$ [4]. In this case

(2.4) $g(\phi X, Y) = -g(X,\phi Y)$

An almost hyperbolic contact metric structure (ϕ,ξ,η,g) on $\bar{M}$ is called trans-hyperbolic Sasakian [6] if and only if

(2.5) $(\bar{\nabla}_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)\phi X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}$

for all X, Y tangents to $\bar{M}$ and α,β are functions on $\bar{M}$. On a trans-hyperbolic Sasakian manifold M, we have

(2.6) $\bar{\nabla}_X\xi = -\alpha(\phi X) + \beta\{X - \eta(X)\xi\}$

a Riemannian metric g and Riemannian connection $\bar{\nabla}$.

Further, an almost contact metric manifold $\bar{M}$ on (ϕ,ξ,η,g) is called nearly trans-hyperbolic Sasakian if [5]

(2.7) $(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\}$

On other hand, a quarter symmetric semi metric connection $\overline{\nabla}$ on M is defined by

(2.8) $\overline{\nabla}_X Y = \overline{\nabla}^*_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi$

Using (2.1), (2.2) and (2.6) in (2.5) and (2.6), we get respectively

(2.9) $(\overline{\nabla}_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)\phi X\}$
$+\beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\} - g(X,Y)\xi - \eta(X)\eta(Y)\xi$

(2.10) $\overline{\nabla}_X\xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\}$

In particular, an almost contact metric manifold $\overline{M}$ on (ϕ,ξ,η,g) is called nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric semi metric connection if

(2.11) $(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \; -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$

Now, let M be a submanifold immersed in $\overline{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connection on M, then the Gauss and
Weingarten formulas for the quarter symmetric semi metric connection are given by

(2.12) $\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$

(2.13) $\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N - \eta(X)\phi N$

for any $X, Y \epsilon TM$ and $V \epsilon T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

(2.14) $g(A_N X, Y) = g(h(X,Y), V)$

For any $x \epsilon M$ and $X \epsilon T_X M$, we write

(2.15) $X = PX + QX$

where $PX \epsilon D$ and $QX \epsilon D^{\perp}$.
Similarly for N normal to M, we have

(2.16) $\phi N = BN + CN$

where BN (respectly CN) is the tangential component (respectly normal component) of ϕN.

Definition. An m dimensional Riemannian submanifold M of $\overline{M}$ is called a CR-submanifold of M if there exists a differentiable distribution $D : x \rightarrow D_X$ on M satisfying the following conditions:

(i) D is invariant, that is $\phi D_X \subset D_X$ for each $x \epsilon M$,

(ii) The complementary orthogonal distribution $D^{\perp}: X \to D_X^{\perp} \subset T_X M$ of D is anti-invariant, that is, $\phi D_X^{\perp} \subset T_X^{\perp} M$ for each $x \in M$. If $\dim D_X^{\perp} = 0$ (respectly $\dim D_X = 0$), then the CR-submanifold is called an invariant (respectly, anti-invariant) submanifold. The distribution D (respectly, $D^{\perp}$) is called the horizontal (respectly, vertical) distribution. Also, the pair $(D, D^{\perp})$ is called ξ -horizontal (respectly, vertical) if $\xi_X \in D_X$_(respectly, $\xi_X \in D_X^{\perp}$).

3. Some Basic Lemmas

Lemma 1. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric semi metric connection. Then

(3.1) $$P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y$$
$$= 2(\alpha - 1)g(X,Y)P\xi - \alpha\eta(Y)\phi PX - \alpha\eta(X)\phi PY - \beta\eta(Y)\phi PX - \beta\eta(X)\phi PY$$
$$-4\eta(X)\eta(Y)P\xi + \phi P\nabla_X Y + \phi P\nabla_Y X$$

(3.2) $$Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y$$
$$= 2Bh(X,Y) + 2(\alpha - 1)g(X,Y)Q\xi - \alpha\eta(Y)\phi QX - \alpha\eta(X)\phi QY$$
$$+\eta(X)QY + \eta(Y)QX - 4\eta(X)\eta(Y)Q\xi$$

(3.3) $$h(X, \phi PY) + h(Y, \phi PX) + D_X^{\perp}\phi QY + D_Y^{\perp}\phi QX$$
$$= \phi Q\nabla_Y X + \phi Q\nabla_X Y + 2Ch(X,Y) - \beta\eta(Y)\phi QX - \beta\eta(X)\phi QY$$

for any $X, Y \in TM$.

Proof. Using (2.4) (2.9), and (2.10) in (2.11) we get

$$(\nabla_X\phi PY) + h(X,\phi PY) - A_{\phi QY}X + D_X^{\perp}\phi QY - \phi(\nabla_X Y) - \phi h(X,Y) - \eta(X)QY$$
$$+\eta(X)\eta(Y)\xi + (\nabla_Y\phi PX) + h(Y,\phi PX) - A_{\phi QX}Y + D_Y^{\perp}\phi QX - \phi(\nabla_Y X)$$
$$-\phi h(Y,X) - \eta(Y)QX + \eta(X)\eta(Y)\xi = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\}$$
$$-\beta\{\eta(X)\phi Y + \eta(Y)\phi X\} - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

Again using (2.15) we get

(3.4) $$P(\nabla_X\phi PY) + P(\nabla_Y\phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y - \phi P\nabla_X Y - \eta(X)QY$$
$$-\phi Q\nabla_X Y - \phi P\nabla_Y X - \phi Q\nabla_Y X + Q(\nabla_X\phi PY) + Q(\nabla_Y\phi PX) + 2\eta(X)\eta(Y)P\xi$$
$$+2\eta(X)\eta(Y)Q\xi - QA_{\phi QY}X - QA_{\phi QX}Y + h(X,\phi PY) + h(Y,\phi PX) + D_X^{\perp}\phi QY$$
$$-\eta(Y)QX + D_Y^{\perp}\phi QX - 2Bh(X,Y) - 2Ch(X,Y) = 2\alpha g(X,Y)P\xi +$$
$$2\alpha g(X,Y)Q\xi$$
$$-\alpha\eta(Y)\phi PX - \alpha\eta(Y)\phi QX - \alpha\eta(X)\phi PY - \alpha\eta(X)\phi QY - \beta\eta(X)\phi PY$$
$$-\beta\eta(X)\phi QY - \beta\eta(Y)\phi PX - \beta\eta(Y)\phi QX - 2\eta(X)\eta(Y)P\xi - 2\eta(X)\eta(Y)Q\xi$$
$$-2g(X,Y)P\xi - 2g(X,Y)Q\xi$$

for any $X, Y \in TM$.

Now equating horizontal, vertical, and normal components in (3.4), we get the desired result.

Lemma 2. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric semi metric connection. Then

(3.5) $$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \\ +\alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \\ -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

(3.6) $$2(\bar{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \\ -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \nabla_X\phi Y + \nabla_Y\phi X + h(X,\phi Y) + h(Y,\phi X) + \\ \phi[X,Y]$$

Proof. From Gauss formula (2.12), we have

(3.7) $$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X)$$

Also we have

(3.8) $$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]$$

From (3.6) and (3.7), we get

(3.9) $$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

Also for nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric semi metric connection, we have

(3.10) $$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \\ \eta(Y)\phi X\} \; -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

Adding (3.9) and (3.10), we get

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \\ +\alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \\ -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

Subtracting (3.9) from (3.10) we get

$$2(\bar{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} \\ -2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \nabla_X\phi Y + \nabla_Y\phi X - h(X,\phi Y) + \\ h(Y,\phi X) + \phi[X,Y]$$

Hence Lemma is proved.

Lemma 3. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\bar{M}$ with a quarter symmetric semi metric connection, then

$$2(\bar{\nabla}_Y\phi)(Z) = \mathrm{A}_{\phi Y}Z - \mathrm{A}_{\phi Z}Y - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \eta(Y)Z + \eta(Z)Y - \phi[Y,Z] \\ +\alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \\ -2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi$$

$$2(\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} - 2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi - A_{\phi Y}Z + A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \phi[Y,Z]$$

for any $Y, Z \epsilon D^{\perp}$.

Proof. From Weingarten formula (2.13), we have

(3.11) $\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \eta(Y)Z + \eta(Z)Y$

Also, we have

(3.12) $\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = (\overline{\nabla}_Y\phi)Z - (\overline{\nabla}_Z\phi)Y + \phi[Y,Z]$

From (3.11) and (3.12), we get

(3.13) $(\overline{\nabla}_Y\phi)Z - (\overline{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi - \eta(Y)Z + \eta(Z)Y - \phi[Y,Z]$

Also for nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric semi metric connection, we have

(3.14) $(\overline{\nabla}_Y\phi)Z + (\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} \; -2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi$

Adding (3.13) and (3.14), we get

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \phi[Y,Z] - \eta(Y)Z + \eta(Z)Y + \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} - 2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi$$

Subtracting (3.13) from (3.14) we get

$$2(\overline{\nabla}_Z\phi)Y = \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} - \beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} - 2\eta(Y)\eta(Z)\xi \; -2g(Y,Z)\xi - A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z + \eta(Y)Z - \eta(Z)Y + \phi[Y,Z]$$

This proves our assertions.

Lemma 4. Let M be a CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric semi metric connection, then

$$2(\overline{\nabla}_X\phi)Y = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \eta(X)Y + \eta(X)\eta(Y)\xi \quad -\nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

$$2(\overline{\nabla}_Y\phi)X = \alpha\{2g(X,Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi + A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \eta(X)Y - \eta(X)\eta(Y)\xi \quad +\nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y]$$

for any $X \epsilon D$and $Y \epsilon D^{\perp}$.

Proof. By using Gauss equation and Weingarten equation for $X \epsilon D$ and $Y \epsilon D^{\perp}$ respectively we get

(3.15) $$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \eta(X) Y + \eta(X)\eta(Y)\xi - \nabla_Y \phi X - h(Y, \phi X)$$

Also, we have

(3.16) $$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi) Y - (\overline{\nabla}_Y \phi) X + \phi[X, Y]$$

From (3.15) and (3.16), we get

(3.17) $$(\overline{\nabla}_X \phi) Y - (\overline{\nabla}_Y \phi) X = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \eta(X) Y + \eta(X)\eta(Y)\xi - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

Also for nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric semi metric connection, we have

(3.18) $$(\overline{\nabla}_X \phi) Y + (\overline{\nabla}_Y \phi) X = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(X)\phi Y + \eta(Y)\phi X\} - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi$$

Adding (3.17) and (3.18), we get

$$2(\overline{\nabla}_X \phi) Y = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi - \mathrm{A}_{\phi Y} X + \nabla_X^{\perp} \phi Y - \eta(X) Y + \eta(X)\eta(Y)\xi - \nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y]$$

Subtracting (3.9) from (3.10) we get

$$2(\overline{\nabla}_Y \phi) X = \alpha\{2g(X, Y)\xi - \eta(Y)\phi X - \eta(X)\phi Y\} - \beta\{\eta(Y)\phi X + \eta(X)\phi Y\} - 2\eta(X)\eta(Y)\xi - 2g(X, Y)\xi + \mathrm{A}_{\phi Y} X - \nabla_X^{\perp} \phi Y + \eta(X) Y - \eta(X)\eta(Y)\xi + \nabla_Y \phi X + h(Y, \phi X) + \phi[X, Y]$$

Hence Lemma is proved.

4. Parallel Distributions

Definition. The horizontal (respectly, vertical) distribution D (respectly, $D^{\perp}$) is said to be parallel [1] with respect to the connection on M if $\nabla_X Y \epsilon D$ (respectly, $\nabla_Z W \epsilon D^{\perp}$) for any vector field $X, Y \epsilon D$ (respectly, $W, Z \epsilon D^{\perp}$).

Proposition 1. Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian manifold $\overline{M}$ with a quarter symmetric semi metric connection. If the horizontal distribution D is parallel, then

(4.1) $h(X, \phi Y) = h(Y, \phi X)$

for all $X,\ Y \epsilon D$.

Proof. Using parallelism of horizontal distribution D, we have

(4.2) $\nabla_X \phi Y \epsilon D,\ \nabla_Y \phi X \epsilon D$ for any $X,\ Y \epsilon D$.

Thus using the fact that $X = QY = 0$ for $Y \epsilon D$, (3.2) gives

(4.3) $Bh(X,Y) = g(X,Y)Q\xi$ for any $X,\ Y \epsilon D$.

Also, since

(4.4) $\phi h(X,Y) = Bh(X,Y) + Ch(X,Y)$,

then

(4.5) $\phi h(X,Y) = g(X,Y)Q\xi + Ch(X,Y)$ for any $X,\ Y \epsilon D$.

Next from (3.3), we have

(4.6) $h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) - 2g(X,Y)Q\xi$,

for any $X,\ Y \epsilon D$. Putting $X = \phi X \epsilon D$ in (4.6), we get

(4.7) $h(\phi X,\phi Y) + h(Y,\phi^2 X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi$

or

(4.8) $h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi$

Similarly, putting $Y = \phi Y \epsilon D$ in (4.6), we get

(4.9) $h(\phi Y,\phi X) - h(X,Y) = 2\phi h(X,\phi Y) - 2g(X,\phi Y)Q\xi$.

Hence from (4.8) and (4.9), we have

(4.10) $\phi h(X,\phi Y) - \phi h(Y,\phi X) = g(X,\phi Y)Q\xi - g(\phi X,Y)Q\xi$

Operating ϕ on both sides of (4.10) and using $\phi\xi = 0$, we get

(4.11) $h(X,\phi Y) = h(Y,\phi X)$

for all $X,\ Y \epsilon D$.

Now, for the distribution $D^{\perp}$, we prove the following proposition.

Proposition 2. Let M be a ξ-vertical CR-submanifold of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric semi metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

(4.12) $A_{\phi Y}Z + A_{\phi Z}Y\ \epsilon\ D^{\perp}$ for any $Y,Z\ \epsilon\ D^{\perp}$.

Proof. Let $,Z\ \epsilon\ D^{\perp}$, then using Gauss and Weingarten formula (2.10), we obtain

(4.13) $-A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \eta(Y)Z + \eta(Y)\eta(Z)\xi - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \eta(Z)Y + \eta(Z)\eta(Y)\xi$
$= \phi\nabla_Y Z + \phi h(Y,Z) + \phi\nabla_Z Y + \phi h(Z,Y) + \alpha\{2g(Y,Z)\xi - \eta(Y)\phi Z - \eta(Z)\phi Y\} \quad -\beta\{\eta(Y)\phi Z + \eta(Z)\phi Y\} - 2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi$

for any $Y,Z\ \epsilon\ D^{\perp}$. Taking inner product with $X \epsilon D$ in (4.13), we get

(4.14) $g\big(A_{\phi Y}Z,X\big) + g\big(A_{\phi Z}Y,X\big) = g(\nabla_Y Z,\phi X) + g(\nabla_Z Y,\phi X)$

If the distribution $D^{\perp}$ is parallel, then $\nabla_Y Z \epsilon D^{\perp}$ and $\nabla_Z Y \epsilon D^{\perp}$, for any $Y,Z\ \epsilon\ D^{\perp}$.
So from (4.14) we get

(4.15) $g\big(A_{\phi Y}Z,X\big) + g\big(A_{\phi Z}Y,X\big) = 0$ or (4.13) $g\big(A_{\phi Y}Z + A_{\phi Z}Y,X\big) = 0$

which is equivalent to

(4.15) $\quad A_{\phi Y}Z + A_{\phi Z}Y \in D^{\perp}$ for any $Y, Z \in D^{\perp}$

and this completes the proof.

Definition : A CR-submanifold M of a nearly trans-hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric semi metric connection is said to be totally geodesic if $h(X, Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$.

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in T^{\perp}M$.

Let $X \in D$ and $Y \in \phi D^{\perp}$. For a mixed totally geodesic ξ-vertical CR-submanifold M of a nearly trans hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric semi metric connection then from (2.9), we have

$$(\bar{\nabla}_X \phi)N = 0$$

Since $\bar{\nabla}_X \phi N = (\bar{\nabla}_X \phi)N + \phi(\bar{\nabla}_X N)$ so that $\bar{\nabla}_X \phi N = \phi(\bar{\nabla}_X N)$.

Hence in view of (2.13), we get

$$\bar{\nabla}_X \phi N = -A_{\phi N}X + \nabla_X^{\perp} \phi N = -\phi A_N X + \phi \nabla_X^{\perp} N$$

As $A_N X \in D$, $\phi A_N X \in D$, so $\phi \nabla_X^{\perp} N = 0$ if and only if $\bar{\nabla}_X \phi N \in D$.

Thus we have the following proposition.

Proposition 3. Let M be a mixed totally geodesic ξ-vertical CR-submanifold of a nearly trans hyperbolic Sasakian Manifold $\bar{M}$ with a quarter symmetric semi metric connection. Then the normal section $N \in \phi D^{\perp}$ is D parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.

5. Conclusion

The notion of CR- submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection investigated which shows that the existence of a parallel distribution relating to ξ-vertical CR-submanifolds of a nearly trans-hyperbolic sasakian manifold with a quarter symmetric semi metric connection. Further I have tried to find the condition under which the distributions required by CR-submanifolds of a nearly trans-hyperbolic Sasakian manifold with a quarter symmetric semi metric connection are parallel are obtained. D-parallel normal section have been also studied.

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