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CONTROL DIFFUSION PROCESSES WITH LIPSCHITZ CONTINUITY OF DRIFTS

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Abstract: Control diffusion processes has been found in a wide field of applications as in stochastic optimal control and in mathematical finance via the theory of hedging and nonlinear pricing theory for imperfect markets. In this paper we discuss the control diffusion process with time and space dependent coefficients and local Lipschitz continuity of the drift. The results show that the controlled process $X_t^{s,\xi,u}$ is independent of control u for a constant.

Keywords: Stochastic Differential Equations, Lipschitz continuity, Control Diffusion Process

1. Introduction

Given a bounded Borel subset $U \subset \mathbb{R}^n$, we denote by $\mathcal{U}$ a set of progressively measurable processes $u = (u_t,\ t \geq 0)$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{P}(u_t \in U) = 1$ for all $t \geq 0$. The elements of $\mathcal{U}$ are called *admissible control processes*. For each control process $(u_t) \in \mathcal{U}$, we consider a stochastic differential equation,

$$\begin{cases} dX_t^{s,\xi,u} = b(t, X_t^{s,\xi,u}, u_t)dt + \sigma(t, X_t^{s,\xi,u}, u_t)dW_t, & t \geq s, \\ X_s = \xi \end{cases} \tag{1}$$

where $X_t^{s,\xi,u} \in \mathbb{R}^d$, and $b : \mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$, and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times n}$ are assigned Lipschitz continuous functions for each $u \in \mathbb{R}^n$. We interpret $X_t = X_t(\omega)$ as the state of the system at time t. By a pathwise solution of this equation, we mean an $(\mathcal{F}_t)$-adapted continuous stochastic process $X_t^{s,x,u}$ satisfying

$$X_t^{s,\xi,u} = \xi + \int_s^t b(r, X_r^{s,\xi,u}, u_r)dr + \int_s^t \sigma(r, X_r^{s,\xi,u}, u_r)dW_r, \quad 0 \leq s \leq t. \tag{2}$$

If the above equation has a unique solution $X_t^{s,\xi,u}$, the process (X_t) is called a controlled process.

This type of problem appears in many applications in insurance and finance. In insurance, Luo [17] consider an optimal dynamic control problem for an insurance company with opportunities of proportional reinsurance and investment. Liang [16] study optimal proportional reinsurance policy of an insurer with a risk process which is perturbed by a diffusion. The closed-form expressions for the policy and the value function are derived in the sense of maximizing the expected utility in the jump-diffusion framework.

Another papers discussing the application of control diffusion process in its application can be seen in [5], [7], [12], [13]. In financial applications, one may refer to [1], [9], [11], [14].

2. Some useful facts

In this section, we review some facts which are important in next section.

Definition 2.1. A continuous function $v : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be *upper semicontinuous* if

$$v(s, x) \geq \limsup_{n \to \infty} v(s_n, x_n)$$

for any $s_n \in [0, T)$ and $x \in \mathbb{R}$ whenever $s_n \to s$ and $x_n \to x$, as $n \to \infty$.
A continuous function $v : [0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be *lower semicontinuous* if

$$v(s, x) \leq \liminf_{n \to \infty} v(s_n, x_n)$$

for any $s, s_n \in [0, T)$ and $x \in \mathbb{R}$ whenever $s_n \to s$ and $x_n \to x$ as $n \to \infty$.

The next lemma is well known.

Lemma 2.1. *Let $\{v^\alpha \; ; \; \alpha \in \mathcal{A}\}$ be a family of lower semicontinuous functions. Then*

$$v(s, x) = \sup_{\alpha \in \mathcal{A}} v^\alpha(s, x)$$

is lower semicontinuous.

Lemma 2.2. Gronwall Lemma
Suppose that the function $F : [0, T] \rightarrow [0, \infty)$ satisfies conditions

$$\int_0^T F(t) dt < \infty \tag{3}$$

and

$$F(t) \leq a + b \int_0^t F(s) ds, \quad t \leq T \tag{4}$$

where $a, b \geq 0$. Then

$$F(t) \leq a e^{bt}, \quad t \leq T \tag{5}$$

The proof of the Gronwall Lemma is well known, one may refer to Dharmawan [7] or Bouchard [5] for the complete proof.

The following is a standing assumption on the functions b and σ appearing in the control system.

Assumption 2.1. For each $T > 0$ there exists a constant $K > 0$ such that for all $u \in U, s \leq T$ and $x, y \in \mathbb{R}^d$

$$|b(s, x, u) - b(s, y, u)| + |\sigma(s, x, u) - \sigma(s, y, u)| \leq K|x - y|. \tag{6}$$

$$|b(s, x, u)| + |\sigma(s, x, u)| \leq K(1 + |x|) \tag{7}$$

It is well known, see for example [11], p.158 or [22], that Assumption 2.1 yields the existence of a unique strong solution $(X_t^{s,\xi,u})$ to (1), for each $s > 0$, each initial condition ξ, and each $u \in U$. Moreover, $(X_t^{s,\xi,u})$ is continuous on $[s, T]$.

Various versions of the next results are well known, see for example the monograph of Krylov [15] or Bouchard [5].

Definition 2.2. Quadratic Variation of Martingales
Let $\delta_n = \max_i (t_{i+1}^n - t_i^n) \ \to \infty$ as $n \to \infty$. The Quadratic variation of a process (X_t) is defined as a limit in probability

$$\langle X \rangle_t = \lim_{\delta \to \infty} \sum_{i=1}^{n} (X_{t_i^n} - X_{t_{i-1}^n})^2. \tag{8}$$

If (X_t) is a martingale, then (X_t^2) is a submartingale. By compensating X_t^2 by an increasing process, it is possible to make it into a martingale. The process which compensates X_t^2 to form a martingale turns out to be the quadratic variation of process X_t.

Theorem 2.1. *If (X_t) is a local martingale, then $\langle X, X \rangle_t$ exists. Moreover $X_t^2 - \langle X, X \rangle_t$ is a local martingale.*

Theorem 2.2. Burkholder-Davis-Gundy
For every $p \geq 1$, there exist two constants c_p and C_p such that, for all continuous local martingales M vanishing at zero,

$$c_p \mathbb{E}\left[\langle M, M \rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right] \leq C_p \mathbb{E}\left[\langle M, M \rangle_\infty^{p/2}\right]$$

where $M_t^* = \sup_{s \leq t} |M_s|$.

3. Results

In this section we prove some results. The results here are not really new, but the proofs are my original works. Another version of the proofs can be seen in [5].

Theorem 3.1. *Let ξ be an $\mathcal{F}_s$-measurable random variable and for $p \geq 2$ such that $\mathbb{E}|\xi|^p < \infty$. Then there exists a constant $K(T,p) > 0$ which is independent of u such that for all $0 \leq s \leq t \leq T$,*

$$\mathbb{E}|X_t^{s,\xi,u}|^p \leq K \mathbb{E}(1 + |\xi|^p). \tag{9}$$

Proof. We define the stopping times

$$\tau_n = \begin{cases} \inf\{t \in [s,T]; |X_t^{s,\xi,u}| \geq n\}, & n \geq 1, \\ \\ T, \ \text{ if } \ \{t \ [s,T]; |X_t^{s,\xi,u}| \geq n\} = \emptyset \end{cases} \tag{10}$$

The stopping times τ_n are well defined since the process $X_t^{s,\xi,u}$ is continuous in $t \in [s,T]$. Then following (2) we have

$$X_{t \wedge \tau_n}^{s,\xi,u} = \xi + \int_s^{t \wedge \tau_n} b(r, X_r^{s,\xi,u}, u_r) dr + \int_s^{t \wedge \tau_n} \sigma(r, X_r^{s,\xi,u}, u_r) dW_r, \quad 0 \leq s \leq t \leq T. \tag{11}$$

Invoking the Burkholder-Davis-Gundy inequalities 2.2 we obtain

$$\begin{aligned}\mathbb{E}|X_{t\wedge\tau_n}|^p &\leq 3^{p-1}\mathbb{E}|\xi|^p + 3^{p-1}\mathbb{E}\left[\int_s^{t\wedge\tau_n}\left|b(r,X_r^{s,\xi,u},u_r)\right|dr\right|^p \\ &+ \frac{1}{2}3^{p-1}\left[\int_s^{t\wedge\tau_n}\mathbb{E}\left[\mathrm{tr}(\sigma\sigma^*)(r,X_r^{s,\xi,u},u_r)\right]dr\right|^{p/2}. \end{aligned} \tag{12}$$

Using the Jensen inequality and Assumption 2.1 we find that

$$\begin{aligned}\mathbb{E}|X_{t\wedge\tau_n}^{s,\xi,u}|^p &\leq 3^{p-1}\mathbb{E}|\xi|^p + (3T)^{p-1}K^p\mathbb{E}\left[\int_s^{t\wedge\tau_n}(1+|X_r^{s,\xi,u}|^p)dr\right] \\ &+ \frac{1}{2}6^{p-1}T^{\frac{p}{2}-1}\mathbb{E}\left[\int_s^{t\wedge\tau_n}(1+|X_r^{s,\xi,u}|^p)dr\right]. \end{aligned} \tag{13}$$

Therefore, there exists a constant $k > 0$ such that

$$\mathbb{E}|X_{t\wedge\tau_n}|^p \leq 3^{p-1}\mathbb{E}|\xi|^p + k\mathbb{E}\int_s^T \left(1+|X_{r\wedge\tau_n}^{s,\xi,u}|^p\right)dr. \tag{14}$$

The function $g_n(t) = \mathbb{E}|X_{t\wedge\tau_n}^{s,\xi,u}|^p$ is integrable on $[0,T]$ by definition of τ_n and

$$g_n(t) \leq 3^{p-1}\mathbb{E}|\xi|^p + k\int_s^t (1+g_n(r))\,dr \quad t \geq s. \tag{15}$$

Therefore, by the Gronwall's inequality (Lemma 2.2) to (15)

$$g_n(t) \leq \left(3^{p-1}\mathbb{E}|\xi|^p + T\right)e^{kt} \quad t \in [s,T].$$

Applying the Fatou Lemma in order to pass with $n \to \infty$ in the above inequality we conclude the proof. □

Theorem 3.2. *Let X_t^n be the solution of the stochastic differential equation*

$$X_t^{s,x_n,u} = \xi^n + \int_s^t b(r,X_r^{s,\xi^n,u},u_r)dr + \int_s^t \sigma(r,X_r^{s,\xi^n,u},u_r)dW_r, \qquad 0 \leq s \leq t \tag{16}$$

Let (X_t) be the solution of (11) and assume that for a certain $p \geq 2$

$$\mathbb{E}\left(|\xi^n|^p + |\xi|^p\right) < \infty, \quad n \geq 1.$$

Then there exists a constant $C(p,K,T)$ which is independent of u such that

$$\mathbb{E}\sup_{t\leq T}|X_t^n - X_t|^p \leq C(p,K,T)\mathbb{E}\left|\xi^n - \xi\right|^p.$$

Proof. We proceed similarly as in the proof of the previous Theorem, hence some details are omitted. Using Lipschitz property of coefficients and 2.2 we obtain

$$
\begin{aligned}
& \mathbb{E} \sup_{t\le T} |X_t^n - X_t|^p \\
\le\ & 3^{p-1}\mathbb{E}|\xi^n - \xi|^p + 3^{p-1}\mathbb{E}\sup_{s\le t\le T}\left|\int_s^t [b(r,X_r^n,u_r) - b(r,X_r,u_r)]\,dr\right|^p \\
\le\ & 3^{p-1}\mathbb{E}|\xi^n - \xi|^p + C_1(p,K)\mathbb{E}\int_s^T |b(r,X_r^n,u_r) - b(r,X_r,u_r)|^p\,dr \\
& \quad + C_1(p,K,T)\mathbb{E}\int_s^T \|\sigma(r,X_r^n,u_r) - \sigma(r,X_r,u_r)\|^p\,dr \\
\le\ & 3^{p-1}\mathbb{E}|\xi^n - \xi|^p + C_1(p,K,T)\mathbb{E}\int_s^T |X_r^n - X_r|^p\,dr.
\end{aligned}
$$

Now we apply the Gronwall inequality (Lemma 2.2) with $g_n(t) = \mathbb{E}\sup_{s\le u\le t}|X_t^n - X_t|^p$ to obtain

$$
\begin{aligned}
\mathbb{E}\sup_{t\le T}|X_t^n - X_t|^p &\le 3^{p-1}\mathbb{E}\,|\xi^n - \xi|^p + C_1(p,K)(T-s)e^{C_1 t} \\
&\le C(p,K,T)\mathbb{E}\,|\xi^n - \xi|^p
\end{aligned}
$$

where $C(p,K,T) = \max(3^{p-1}, C_1(p,K)(T-s)e^{C_1 T})$. □

Theorem 3.3. *Let $X_t^{s_n,\xi,u}$, where $0 \le s_n \le t \le T$ be a solution of the stochastic differential equation*

$$
X_{t\wedge T}^{s_n,\xi,u} = \xi + \int_{s_n}^{t\wedge T} b(r, X_r^{s_n,\xi,u}, u_r)dr + \int_{s_n}^{t\wedge T} \sigma(r, X_r^{s_n,\xi,u}, u_r)dW_r, \quad s_n \le t \le T \tag{17}
$$

Then for all $p \ge 2$ there exists a constant $C(T,p)$ which is independent of u and such that

$$
\mathbb{E}\sup_{\bar{s}\le t\le T}\left|X_t^{s_n,\xi,u} - X_t^{s,\xi,u}\right|^p \le C(T,p)|s_n - s|^{p/2}
$$

where $\bar{s} = \max(s, s_n)$.

Proof. We will prove the Theorem assuming that $s_n < s$. The case of $s_n > s$ is completely analogous and omitted. For simplicity we assume also that the drift $b = 0$. Then we have

$$
\begin{aligned}
X_t^{s_n,\xi,u} &= \xi + \int_{s_n}^t \sigma(r, X_r^{s_n,\xi,u}, u_r)dW_r, \quad s_n \le t \le T, \\
X_t^{s,\xi,u} &= \xi + \int_{s}^t \sigma(r, X_r^{s,\xi,u}, u_r)dW_r, \quad s \le t \le T.
\end{aligned}
$$

Let $X_t^n = X_t^{s_n,x,u}$ and $X_t = X_t^{s,x,u}$. Then for $s \leq t'$ and invoking again 2.2 we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\leq t'}\|X_t^n &- X_t\|^p \\
&\leq C_1(T,p)\mathbb{E}\left(\int_{s_n}^{s}\|\sigma(r,X_r^n,u_r)\|^2\,dr\right)^{p/2} + \\
&\quad C_1(T,p)\mathbb{E}\left(\int_{s}^{t'}\|\sigma(r,X_r^n,u_r)-\sigma(r,X_r,u_r)\|^2\,dr\right)^{p/2} \\
&\leq C(T,p)|s_n - s|^{p/2} + C(T,p)\mathbb{E}\int_s^{t'}\|X_r^n - X_r\|^p dr.
\end{aligned}$$

Now we apply the Gronwall's inequality (Lemma 2.2) with $g(t) = \mathbb{E}\sup_{s\leq t\leq t'}\|X_t^n - X_t\|^p$. We have

$$\mathbb{E}\sup_{t\leq T}\|X_t^n - X_t\|^p \leq C_1(T,p)|s_n - s|^{p/2} + C_1(T,p)(s_n-s)^{p/2}e^{C_1(t'-s)}.$$

Choose $C(T,p) = \max(C_1(T,p), C_1(T,P)e^{C_1(t'-s)})$; then

$$\mathbb{E}\sup_{t\leq T}\|X_t^n - X_t\|^p \leq C(T,p)|s_n - s|^{p/2}.$$

□

Theorem 3.4. *Let Assumption 2.1 hold. For each $p \geq 1$, $T > 0$, $t \geq s_2 > s_1 > 0$,*

$$\mathbb{E}\sup_{s_2\leq t\leq T}|X_t^{s_2,x_2,u} - X_t^{s_1,x_2,u}|^p \leq C_1(|x_2-x_1|^p + |s_2-s_1|^{p/2}),$$

where C_1, is independent of u, s, s_n, ξ.

Proof. The proof is provided for $b = 0$. The general case does not lead to any additonal difficulties. Let $X_t^1 = X_t^{s_1,x_1,u}$, $X_t^2 = X_t^{s_2,x_2,u}$. Then

$$\begin{aligned}
\mathbb{E}\sup_{s_2\leq t\leq t'}\|X_t^2 - X_t^1\|^p &\leq 3^{p-1}|x_2-x_1|^p + 3^{p-1}\mathbb{E}\left|\int_{s_1}^{s_2}\sigma(r,X_r^1,u_r)dW_r\right|^p \\
&\quad + 3^{p-1}\mathbb{E}\sup_{s_2\leq t\leq t'}\left|\int_{s_2}^{t}(\sigma(s,X_s^2,u_s)-\sigma(s,X_s^1,u_s))dW_s\right|^p.
\end{aligned}$$

Then, using the Burkholder-Davis-Gundy inequality (Theorem 2.2) we obtain

$$\begin{aligned}
\mathbb{E}\sup_{s_2\leq t\leq t'}\|X_t^2 - X_t^1\|^p &\leq 3^{p-1}|x_2-x_1|^p + 3^{p-1}C_p\mathbb{E}\left(\int_{s_1}^{s_2}|\sigma(r,X_r^1,u_r)|^2dr\right)^{p/2} \\
&\quad + 3^{p-1}C_p\mathbb{E}\left(\int_{s_2}^{t'}|\sigma(s,X_s^2,u_s)-\sigma(s,X_s^1,u_s)|^2ds\right)^{p/2}.
\end{aligned}$$

Using the assumptions on σ and the Jensen inequality we obtain

$$\begin{aligned}\mathbb{E}\sup_{t\le T}\|X_t^2 - X_t^1\|^p &\le 3^{p-1}|x_2 - x_1|^p + c_1|s_2 - s_1|^{\frac{p}{2}-1}\int_{s_1}^{s_2}\left(1 + \left|X_r^1, u_s)\right|^p\right)dr \\ &\quad + c_2 T^{p/2}\mathbb{E}\int_{s_2}^{t'}|X_r^2 - X_r^1|^p dr,\end{aligned}$$

where $c_1, c_2 > 0$ are constants independent of s_1, s_2, x_1, x_2, u. Let $F(t) = \mathbb{E}\sup_{s_2\le r\le t}\|X_r^2 - X_r^1\|^p$. The above inequality and Theorem 3.1 yield

$$F(t) \le 3^{p-1}|x_2 - x_1|^p + c_3|s_2 - s_1|^{p/2} + c_4\int_{s_2}^{t}F(r)dr,$$

for some positive constants c_3, c_4 that are independent of s_1, s_2, u. Moreover, if $|x_1|, |x_1| \le R$ then c_3, c_4 depend on R only but not on specific values of x_1 and x_2. By the Gronwall inequality (Lemma 2.2) we have

$$F(t) \le 3^{p-1}|x_2 - x_1|^p + c_3|s_2 - s_1|^{p/2} + c_4|s_2 - s_1|^{p/2}e^{C_4(t-s_2)}.$$

Let $C = \max(3^{p-1}, C_3, C_4 e^{C_1(t-s2)})$. Then

$$\mathbb{E}\sup_{t\le T}\|X_t^2 - X_t^1\|^p \le C(|x_2 - x_1|^p + |s_2 - s_1|^{p/2}).$$

□

Theorem 3.1 - 3.4 are important to show the smoothness of value functions arising in pricing barrier options which appear in non-convex payoff functions.

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