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Model-Check Based on Residual Partial Sums Process of Heteroscedastic spatial Linear Regression Models

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Abstract: It is common in practice to evaluate the correctness of an assumed linear regression model by conducting a model-check method in which the residuals of the observations are investigated. In the asymptotic context instead of observing the vector of the residuals directly, one investigates the partial sums of the observations. In this paper we derive a functional central limit theorem for a sequence of residual partial sums processes when the observations come from heteroscedastic spatial linear regression models. Under a mild condition it is shown that the limit process is a function of Brownian sheet. Several examples of the limit processes are also discussed. The limit theorem is then applied in establishing an asymptotically Kolmogorov type test concerning the adequacy of the fitted model. The critical regions of the test for finite sample sizes are constructed by Monte Carlo simulation.

Keywords: heteroscedastic linear regression model, least squares residual, partial sums process, Brownian sheet, asymptotic model-check.

1. Introduction

Let us consider an experiment performed under $n \times n$ experimental conditions taken from a regular lattice given by

$$\Xi_n := \{(\ell/n, k/n) : \ 1 \leq \ell, k \leq n\}, \ n \in \mathbb{N}.$$

Without loss of generality we consider the unit square $\mathbf{I} := [0,1] \times [0,1]$ as an experimental region instead of any compact subset of $\mathbb{R}^2$. For convenience we take the observations carried out in Ξ_n row-wise initializing at the point $(1/n, 1/n)$ and put them together in an $n \times n$ matrix $\mathbf{Y}(\Xi_n) := (Y_{\ell k})_{k=1,\ell=1}^{n,\ n} \in \mathbb{R}^{n\times n}$, where the observation in the point $(\ell/n, k/n)$ is denoted by $Y_{\ell k}$, $1 \leq \ell, k \leq n$. Consequently, we have a sequence of observable random matrices $(\mathbf{Y}(\Xi_n))_{n\geq 1} \subset \mathbb{R}^{n\times n}$. As usual we furnish the vector space $\mathbb{R}^{n\times n}$ with the Euclidean inner product

$$\langle \mathbf{A}, \mathbf{B}\rangle_{\mathbb{R}^{n\times n}} := trace(\mathbf{A}^\top \mathbf{B}), \ \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}.$$

Let $f_1, \ldots, f_p : \mathbf{I} \to \mathbb{R}$ be known, real-valued regression functions defined on $\mathbf{I}$. For a real-valued function f defined on $\mathbf{I}$, let $f(\Xi_n) := (f(\ell/n, k/n))_{k=1,\ell=1}^{n,\ n} \in \mathbb{R}^{n\times n}$. Our aim is to construct an asymptotic test procedure for the hypothesis

$$H_0 : \ \mathbf{Y}(\Xi_n) = \sum_{i=1}^{p} \beta_i f_i(\Xi_n) + \mathbf{E}_n \text{ vs.} H_1 : \ \mathbf{Y}(\Xi_n) = g(\Xi_n) + \mathbf{E}_n, \tag{1}$$

where $(\beta_1, \ldots, \beta_p)^\top =: \beta \in \mathbb{R}^p$ is a vector of unknown parameters, $\mathbf{E}_n := (\varepsilon_{\ell k})_{k=1,\ell=1}^{n,\ n}$ is an $n \times n$ random matrix whose components are independent, real-valued random variables $\varepsilon_{\ell k}$, $1 \leq \ell, k \leq$

n, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having mean 0 and variance $\sigma^2_{\ell k}$, $0 < \sigma^2_{\ell k} < \infty$, $1 \le \ell, k \le n$, and $g : \mathbf{I} \to \mathbb{R}$ is the unknown true regression function. Thus, under null-hypothesis we consider a *heteroscedastic* linear model, while under the alternative we assume a non-parametric *heteroscedastic* regression model. It is worth mentioning that under H_0 and H_1 we need not to assume any specific distribution for the random errors $\varepsilon_{\ell k}$, $1 \le \ell, k \le n$. Under the assumption $f_1(\Xi_n), \ldots, f_p(\Xi_n)$ are linearly independent in $\mathbb{R}^{n\times n}$, the corresponding matrix of least squares residuals of the observations under H_0 is given by

$$\mathbf{R}_n := (r_{\ell k})_{k=1,\ell=1}^{n,\ n} = \mathbf{E}_n - \sum_{i=1}^{p} \frac{\langle f_i(\Xi_n), \mathbf{E}_n\rangle_{\mathbb{R}^{n\times n}} f_i(\Xi_n)}{\langle f_i(\Xi_n), f_i(\Xi_n)\rangle_{\mathbb{R}^{n\times n}}}.$$

Recently, for a fixed $n \ge 1$, MacNeill and Jandhyala [7], and Xie and MacNeill [11] define an operator $T_n : \mathbb{R}^{n\times n} \mapsto \mathcal{C}(\mathbf{I})$, given by

$$\begin{aligned}\mathbf{T}_n(\mathbf{A})(z_1, z_2) := & \sum_{k=1}^{[nz_2]} \sum_{\ell=1}^{[nz_1]} a_{\ell k} \\ & + (nz_1 - [nz_1]) \sum_{\ell=1}^{[nz_2]} a_{[nz_1]+1,\ell} + (nz_2 - [nz_2]) \sum_{k=1}^{[nz_1]} a_{k,[nz_2]+1} \\ & + (nz_1 - [nz_1])(nz_2 - [nz_2]) a_{[nz_1]+1,[nz_2]+1},\ (z_1, z_2) \in \mathbf{I},\end{aligned}$$

for every $\mathbf{A} = (a_{\ell k})_{k=1,\ell=1}^{n,n}$, where $[t] := \max\{n \in \mathbb{N} : n \le t\}$, $t \in \mathbb{R}$ and $\mathbf{T}_n(\mathbf{A})(t, s) = 0$, if $t = 0$ or $s = 0$. Here $\mathcal{C}(\mathbf{I})$ is the space of continuous functions on $\mathbf{I}$ furnished with the supremum norm. By the operator $\mathbf{T}_n$, the matrix of the least squares residuals is induced into a stochastic process $\{\mathbf{T}_n(\mathbf{R}_n)(t, s) : (t, s) \in \mathbf{I}\}$ having sample paths in $\mathcal{C}(\mathbf{I})$. Let us call this process residual partial sums process. It is common in practice to test (1) by investigating a functional of the residual partial sums process such as a Kolmogorov type statistic, defined by $K_n := \max_{1\le \ell,k\le n} \mathbf{T}_n(\mathbf{R}_n)(\ell/n, k/n)$. Therefore in order to establish this test problem we need to investigate the limit process of the sequence $\{\mathbf{T}_n(\mathbf{R}_n)(t, s) : (t, s) \in \mathbf{I}\}_{n\ge 1}$ under H_0 as well as under H_1. In MacNeill and Jandhyala [7] and in Xie and MacNeill [11] the limit process of this sequence was derived explicitly in which homoscedasticity was assumed, i.e. $\sigma^2_{\ell k} = \sigma^2$, for $1 \le \ell, k \le n$. It was shown therein that under the condition of the regression functions are continuously differentiable, the limit process is a complicated function of the Brownian sheet. In Somayasa [10] the limit process of such a sequence was also derived by generalizing the approach of Bischoff [4] from one to higher dimensional case. In contrast to the result of MacNeill and Jandhyala [7] and Xie and MacNeill [11], Somayasa [10] got the structure of the limit process as a projection of the Brownian sheet onto its reproducing kernel Hilbert space. In this paper we establish the limit process of the heteroscedastic linear regression model defined above, see Section 2. In Section 3 we discuss examples of the limit process corresponding to polynomial models. In Section 4 we construct the critical region of the Kolmogorov type test.

2. Residual partial sums limit process

In the sequel we characterize the heteroscedasticity of the regression model by defining a function $h : \mathbf{I} \to \mathbb{R}_{>0}$, such that $\sigma^2_{\ell k} = h(\ell/n, k/n)$, $1 \le \ell, k \le n$, $n \in \mathbb{N}$, where h is assumed to be a function of bounded variation in the sense of Vitali, see Clarkson and Adams [5].

Definition 2.1. *A stochastic process $\{B_h(t, s) : (t, s) \in \mathbf{I}\}$ is called a h-Brownian sheet on $\mathcal{C}(\mathbf{I})$, if*

1. $B_h(t,s) = 0$ *almost surely (a.s.), if* $t = 0$ *or* $s = 0$.
2. For *every rectangle* $[t_1,t_2]\times[s_1,s_2] \subset \mathbf{I}$, $0 \le t_1 \le t_2 \le 1$, $0 \le s_1 \le s_2 \le 1$,

$$\Delta_{[t_1,t_2]\times[s_1,s_2]}B_h \sim \mathcal{N}\left(0, \int_{[t_1,t_2]\times[s_1,s_2]} h\ d\lambda_{\mathbf{I}}\right),$$

where $\Delta_{[t_1,t_2]\times[s_1,s_2]}B_h := B_h(t_2,s_2) - B_h(t_1,s_2) - B_h(t_2,s_1) + B_h(t_1,s_1)$, and $\lambda_{\mathbf{I}}$ is the Lebesgue measure on $\mathbf{I}$. Random variable $\Delta_{[t_1,t_2]\times[s_1,s_2]}B_h$ is called the increment of B_h over $[t_1,t_2]\times[s_1,s_2]$.
3. *For any two rectangles* $\mathbf{I}_1 \subset \mathbf{I}$, $\mathbf{I}_2 \subset \mathbf{I}$ *with* $\mathbf{I}_1 \cap \mathbf{I}_2 = \emptyset$, $\Delta_{\mathbf{I}_1}B_h$ *and* $\Delta_{\mathbf{I}_2}B_h$ *are mutually independent.*

We refer the reader to MacNeill and et al. [?] for the existence of such a process. In case h is a constant function, B_h is the Brownian sheet whose existence has been studied by Yeh [12], Kuelbs [6], and Park [9]. As a consequent of Definition 2.1, the covariance function of B_h is given by

$$K_{B_h}(t_1,s_1;t_2,s_2) := Cov(B_h(t_1,s_1), B_h(t_2,s_2)) = \int_{[0,t_1\wedge t_2]\times[0,s_1\wedge s_2]} h\ d\lambda_{\mathbf{I}},$$

$(t_1,s_1),(t_2,s_2) \in \mathbf{I}$, where $x \wedge y$ stands for the minimum between x and y.

Theorem 2.1. *Let* $(\mathbf{E}_n)_{n\ge1}$, $\mathbf{E}_n := (\varepsilon_{\ell k})_{k=1,\ell=1}^{n,n}$ *be a sequence of* $n\times n$ *random matrix such that* $\varepsilon_{\ell k}$ *are mutually independent with* $\mathbb{E}(\varepsilon_{\ell k}) = 0$ *and* $Var(\varepsilon_{\ell k}) = h(\ell/n, k/n)$, $1 \le \ell, k \le n$, $n \ge 1$. *Then* $\frac{1}{n}\mathbf{T}_n(\mathbf{E}_n) \xrightarrow{\mathcal{D}} B_h$, *as* $n\to\infty$, *in* $\mathcal{C}(\mathbf{I})$. *Here* $\xrightarrow{\mathcal{D}}$ *stands for the convergence in distribution (weakly), see Billingsley [2], p. 23.*

Proof. See MacNeill and et al. [?]. □

Theorem 2.2. *Let* $f_1,\dots,f_p$ *be continuous and have bounded variation in the sense of Hardy (Clarkson and Adams [5]) on* $\mathbf{I}$. *If* $f_1,\dots,f_p$ *are linearly independent in* $L_2(\mathbf{I},\lambda_{\mathbf{I}})$, *where* $L_2(\mathbf{I},\lambda_{\mathbf{I}})$ *is the Hilbert space of squared integrable functions on* $\mathbf{I}$ *with respect to* $\lambda_{\mathbf{I}}$, *then*

$$\frac{1}{n}\mathbf{T}_n(\mathbf{R}_n) \xrightarrow{\mathcal{D}} B_{h,\tilde{\mathbf{f}}},\ as\ n\to\infty,\ \ in\ \mathcal{C}(\mathbf{I}),$$

where

$$B_{h,\tilde{\mathbf{f}}}(t,s) := B_h(t,s) - \left(\int_{[0,t]\times[0,s]} \tilde{\mathbf{f}}^\top\ d\lambda_{\mathbf{I}}\right)\mathbf{W}^{-1}\left(\int_{\mathbf{I}}^{(R)} \tilde{\mathbf{f}}\ dB_h\right),\ (t,s)\in\mathbf{I},$$

$$\int_{[0,t]\times[0,s]} \tilde{\mathbf{f}}^\top\ d\lambda_{\mathbf{I}} := \left(\int_{[0,t]\times[0,s]} f_1\ d\lambda_{\mathbf{I}},\dots,\int_{[0,t]\times[0,s]} f_p\ d\lambda_{\mathbf{I}}\right)$$

$$\int_{\mathbf{I}}^{(R)} \tilde{\mathbf{f}}\ dB_h := \left(\int_{\mathbf{I}}^{(R)} f_1\ dB_h,\dots,\int_{\mathbf{I}}^{(R)} f_p\ dB_h\right)^\top.$$

Here $\mathbf{W} := \left(\int_{\mathbf{I}} f_i f_j\ d\lambda_{\mathbf{I}}\right)_{i=1,j=1}^{p,p} \in \mathbb{R}^{p\times p}$ *is invertible. Furthermore* $B_{h,\tilde{\mathbf{f}}}$ *is a process with the covariance function given by*

$$\begin{aligned}
K_{B_{h,\tilde{\mathbf{f}}}}(t,s;t',s') :=&\ Cov\left(B_{h,\tilde{\mathbf{f}}}(t,s), B_{h,\tilde{\mathbf{f}}}(t',s')\right)\\
= \int_{[0,t\wedge t'][0,s\wedge s']} h\ d\lambda_{\mathbf{I}} &- \left(\int_{[0,t']\times[0,s']} \tilde{\mathbf{f}}^\top\ d\lambda_{\mathbf{I}}\right)\mathbf{W}^{-1}\left(\int_{[0,t]\times[0,s]} \tilde{\mathbf{f}}h\ d\lambda_{\mathbf{I}}\right)\\
&- \left(\int_{[0,t]\times[0,s]} \tilde{\mathbf{f}}^\top\ d\lambda_{\mathbf{I}}\right)\mathbf{W}^{-1}\left(\int_{[0,t']\times[0,s']} \tilde{\mathbf{f}}h\ d\lambda_{\mathbf{I}}\right)\\
&+ \left(\int_{[0,t]\times[0,s]} \tilde{\mathbf{f}}^\top\ d\lambda_{\mathbf{I}}\right)\mathbf{W}^{-1}\ \left(\int_{\mathbf{I}} f_i f_j h\ d\lambda_{\mathbf{I}}\right)_{i=1,j=1}^{p,\ p}\mathbf{W}^{-1}\left(\int_{[0,t']\times[0,s']} \tilde{\mathbf{f}}\ d\lambda_{\mathbf{I}}\right).
\end{aligned}$$

Here and in the sequel $\int^{(R)}$ *denotes Riemann-Stieltjes integral, see Young [13] and Somayasa [10], p. 115.*

Proof. The proof of Theorem 2.2 in Bischoff [3] and the result of Bischoff [4] can be extended to the case of higher experimental regions. □

3. Examples

In this section we discuss several examples of the residual partial sums limit processes of constant, first-order and second-order regression models.

3.1. Constant regression model

As a simple case, we consider a constant model, i.e. $\mathbf{Y}(\Xi_n) = \beta f_1(\Xi_n) + \mathbf{E}_n$, where β is an unknown parameter and $f_1(t,s) = 1$, for $(t,s) \in \mathbf{I}$. Then the residual partial sums limit process of this model is given by

$$B_{h,\tilde{\mathbf{f}}_0}(t,s) := B_h(t,s) - tsB_h(1,1),\ (t,s) \in \mathbf{I},$$

which is the standard Brownian bridge when h is constant, see e.g. McNeill and Jandhyala [7] and Somayasa [10], p. 20.

3.2. First order regression model

Let us consider a first-order regression model

$$\mathbf{Y}(\Xi_n) = \beta_1 f_1(\Xi_n) + \beta_2 f_2(\Xi_n) + \beta_3 f_3(\Xi_n) + \mathbf{E}_n,$$

where β_1, β_2 and β_3 are unknown parameters, $f_1(t,s) = 1$, $f_2(t,s) = t$ and $f_3(t,s) = s$, for $(t,s) \in \mathbf{I}$. Associated to this model we have

$$\mathbf{W} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1/3 & 1/4 \\ 1/2 & 1/4 & 1/3 \end{pmatrix} \text{ and } \mathbf{W}^{-1} = \begin{pmatrix} 7 & -6 & -6 \\ -6 & 12 & 0 \\ -6 & 0 & 12 \end{pmatrix}.$$

Then the residual partial sums limit process of this model is given by

$$\begin{aligned} B_{h,\tilde{\mathbf{f}}_1}(t,s) :=& B_h(t,s) - (7ts - 3t^2s - 3ts^2)B_h(1,1) \\ & - (-6ts + 6t^2s)\left(B_h(1,1) - \int_{[0,1]} B_h(t,1)dt\right) \\ & - (-6ts + 6ts^2)\left(B_h(1,1) - \int_{[0,1]} B_h(1,s)ds\right),\ (t,s) \in \mathbf{I}. \end{aligned}$$

3.3. Second order regression model

For the third example we consider a second-order polynomial model

$$\mathbf{Y}(\Xi_n) = \beta_1 f_1(\Xi_n) + \beta_2 f_2(\Xi_n) + \beta_3 f_3(\Xi_n) + \beta_4 f_4(\Xi_n) + \beta_5 f_5(\Xi_n) + \beta_6 f_6(\Xi_n) + \mathbf{E}_n,$$

where β_1, β_2, β_3, β_4, β_5 and β_6 are unknown parameters, $f_1(t,s) = 1$, $f_2(t,s) = t$, $f_3(t,s) = s$, $f_4(t,s) = t^2$, $f_5(t,s) = ts$, $f_6(t,s) = s^2$, for $(t,s) \in \mathbf{I}$. Accordingly the matrix $\mathbf{W}$ and $\mathbf{W}^{-1}$ are given by

$$\mathbf{W} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/3 & 1/4 & 1/3 \\ 1/2 & 1/3 & 1/4 & 1/4 & 1/6 & 1/6 \\ 1/2 & 1/4 & 1/3 & 1/6 & 1/6 & 1/4 \\ 1/3 & 1/4 & 1/6 & 1/5 & 1/8 & 1/9 \\ 1/4 & 1/6 & 1/6 & 1/8 & 1/9 & 1/8 \\ 1/3 & 1/6 & 1/4 & 1/9 & 1/8 & 1/5 \end{pmatrix},$$

$$\mathbf{W}^{-1} = \begin{pmatrix} 26 & -54 & -54 & 30 & 36 & 30 \\ -54 & 228 & 36 & -180 & -72 & 0 \\ -54 & 36 & 228 & 0 & -72 & -180 \\ 30 & -180 & 0 & 180 & 0 & 0 \\ 36 & -72 & -72 & 0 & 144 & 0 \\ 30 & 0 & -180 & 0 & 0 & 180 \end{pmatrix}.$$

Let $y_1, y_2, y_3, y_4, y_5, y_6 : \mathbf{I} \to \mathbb{R}$ be functions of $\mathbf{I}$ defined by $y_1(t,s) := 26ts - 27t^2s - 27ts^2 + 10t^3s + 9t^2s^2 + 10ts^3$, $y_2(t,s) := -54ts + 114t^2s + 18ts^2 - 60t^3s - 18t^2s^2 - 60ts^3$, $y_3(t,s) := -54ts + 18t^2s + 114ts^2 - 18t^2s^2 - 60ts^3$, $y_4(t,s) := 30ts - 90t^2s + 60t^3s$, $y_5(t,s) := 36ts - 36t^2s - 36ts^2 + 36t^2s^2$ and $y_6(t,s) := 30ts - 90ts^2 + 60ts^3$. The residual partial sums limit process of this model can be expressed by

$$\begin{aligned} B_{h,\tilde{\mathbf{f}}_2}(t,s) &:= B_h(t,s) - y_1(t,s)B_h(1,1) \\ &- y_2(t,s)\left(B_h(1,1) - \int_{[0,1]} B_h(t,s)dt\right) \\ &- y_3(t,s)\left(B_h(1,1) - \int_{[0,1]} B_h(1,s)ds\right) \\ &- y_4(t,s)\left(B_h(1,1) - \int_{[0,1]} 2B_h(t,1)tdt\right) \\ &- y_5(t,s)\left(B_h(1,1) - \int_{[0,1]} B_h(t,1)dt - \int_{[0,1]} B_h(1,s)ds + \int_{[0,1]} B_h(t,s)dtds\right) \\ &- y_6(t,s)\left(B_h(1,1) - \int_{[0,1]} 2B_h(1,s)sds\right), \ (t,s) \in \mathbf{I}. \end{aligned}$$

4. Kolmogorov type test

Kolmogorov type test for Hypotheses (1) is a test based on the statistic $K_{n,\mathbf{f}} := \max_{0\le \ell,k \le n} \frac{1}{n}\sum_{i=0}^{l}\sum_{j=0}^{k} r_{ij}$. We put $r_{ij} = 0$ if $i = 0$ or $j = 0$. We note that by the property of the partial sums, it holds $K_{n,\mathbf{f}} = \sup_{0\le t,s\le 1} \frac{1}{n}\mathbf{T}_n(\mathbf{R}_n)(t,s)$

Theorem 4.1. *For a fixed $\alpha \in (0,1)$, let $\tilde{c}_\alpha$ be the α-quantile of $\sup_{0\le t,s\le 1} B_{h,\tilde{\mathbf{f}}}(t,s)$, i.e. a constant such that $\mathbb{P}\left\{\sup_{0\le t,s\le 1} B_{h,\tilde{\mathbf{f}}}(t,s) \le \tilde{c}_\alpha\right\} = \alpha$. Then an asymptotically size α test based on $K_{n,\mathbf{f}}$ is given by*

$$\textit{reject } H_0 \textit{ if and only if } K_{n,\mathbf{f}} \ge \tilde{c}_{1-\alpha}.$$

Proof. Let $\chi \subset \mathbb{R}^{n\times n}$ be the sample space of the model. We define a sequence of non randomized test $(\delta_n)_{n\geq 1}$, where $\delta_n : \chi \to \{0,1\}$, such that for $\mathbf{Y}_n \in \chi$,

$$\delta_n(\mathbf{Y}_n) := \mathbf{1}_{\left\{\mathbf{Y}_n : \sup_{0\leq t,s\leq 1} \frac{1}{n}\mathbf{T}_n\left(\mathbf{Y}_n - \mathbf{X}_n(\mathbf{X}_n^\top\mathbf{X}_n)^{-1}\mathbf{X}_n^\top\mathbf{Y}_n\right)\geq \tilde{c}_{1-\alpha}\right\}},$$

where $\mathbf{1}_A$ is the indicator function of A. Then by Theorem 2.1 and by the continuity of supremum function, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_0(\delta_n) &= \mathbb{P}\left\{\sup_{0\leq t,s\leq 1} \frac{1}{n}\mathbf{T}_n\left(\mathbf{E}_n - \mathbf{X}_n(\mathbf{X}_n^\top\mathbf{X}_n)^{-1}\mathbf{X}_n^\top\mathbf{E}_n\right) \geq \tilde{c}_{1-\alpha}\right\}\\ &= \mathbb{P}\left\{\sup_{0\leq t,s\leq 1} B_{h,\tilde{\mathbf{f}}}(t,s) \geq \tilde{c}_{1-\alpha}\right\} = \alpha,\end{aligned}$$

where $\mathbb{E}_0$ is the expectation operator under H_0. The proof is complete because the expression $\lim_{n\to\infty}\mathbb{E}_0(\delta_n) = \alpha$ holds uniformly under H_0. □

Since the quantile $\tilde{c}_{(1-\alpha)}$ of $\sup_{0\leq t,s\leq 1} B_{h,\tilde{\mathbf{f}}}(t,s)$ can not be calculated analytically we approximate the finite sample size quantile of $K_{n,\mathbf{f}}$ by Monte Carlo simulations generated according to following algorithm.

step 1: Fix $n_0 \in \mathbb{N}$.

step 2: Generate M i.i.d. pseudo random matrices $\mathbf{E}_{n_0}^{(j)} := (\varepsilon_{\ell kj})_{k=1,\ell=1}^{n_0,\ n_0}$, with independent components generated from $\mathcal{N}\left(0, h(\ell/n_0, k/n_0)\right)$ random variables, $1 \leq \ell, k \leq n_0$, $j = 1,\ldots,M$.

step 4: Calculate the matrix of residuals $\mathbf{R}_{n_0}^{(j)}$ by the equation

$$\mathbf{R}_{n_0}^{(j)} = \mathbf{E}_{n_0}^{(j)} - \sum_{i=1}^{p} \frac{\langle f_i(\Xi_{n_0}), \mathbf{E}_{n_0}^{(j)}\rangle_{\mathbb{R}^{n_0\times n_0}} f_i(\Xi_{n_0})}{\langle f_i(\Xi_{n_0}), f_i(\Xi_{n_0})\rangle_{\mathbb{R}^{n_0\times n_0}}}.$$

step 5: Calculate the statistic $K_{n_0,\mathbf{f}}^{(j)} := \max_{0\leq k,\ell\leq n_0} \mathbf{T}_{n_0}\left(\mathbf{R}_{n_0}^{(j)}\right)(\ell/n_0, k/n_0)$.

step 6: Calculate the simulated $(1-\alpha)$-quantiles of $\sup_{0\leq t,s\leq 1} B_{h,\tilde{\mathbf{f}}}(t,s)$: Let $K_{n_0,\mathbf{f}}^{(M:j)}$ be the j'th smallest observation, i.e. $K_{n_0,\mathbf{f}}^{(M:1)} \leq \ldots \leq K_{n_0,\mathbf{f}}^{(M:j)} \leq K_{n_0,\mathbf{f}}^{(M:j+1)} \leq \ldots \leq K_{n_0,\mathbf{f}}^{(M:M)}$, then the simulated $(1-\alpha)$-quantile is given by

$$\tilde{c}_{(1-\alpha)} = \begin{cases} K_{n_0,\mathbf{f}}^{(M:M(1-\alpha))}, & \text{if } M(1-\alpha) \in \mathbb{N},\\ \\ K_{n_0,\mathbf{f}}^{(M:[M(1-\alpha)]+1)}, & \text{otherwise,}\end{cases}$$

where $[M(1-\alpha)] = \max\{k \in \mathbb{N} : k \leq M(1-\alpha)\}$.

The simulation results obtained by using the statistical software package R 2.0.1 are presented in Table 1 for $\alpha = 0.005$, 0.010, 0.025, 0.050, 0.100, 0.150, 0.200, 0.250, 0.360 and 0.500, with the corresponding sample size $n_0 = 30$ and the number of replications $M = 10^6$.

5. Concluding remark

This paper discusses an extension of some existing functional central limit theorems for least squares residual partial sums processes of spatial linear regression model, where the experimental design is restricted to a regular lattice. In practice such a type of experimental design is sometimes difficult to be realized because of economic, technical or ecological reasons. In forthcoming paper the consideration will be extended to the problem of arbitrary experimental design by applying the uniform central limit theorem for set-indexed partial-sum processes with finite variance of Alexander and Pyke [1].

Models	$\tilde{c}_{0.5000}$	$\tilde{c}_{0.6500}$	$\tilde{c}_{0.7500}$	$\tilde{c}_{0.8000}$	$\tilde{c}_{0.8500}$
Constant	0.3740	0.4409	0.4839	0.5139	0.5200
First order	0.3691	0.3963	0.4101	0.4179	0.4306
Second order	0.3295	0.3674	0.3873	0.3982	0.4233
Models	$\tilde{c}_{0.9000}$	$\tilde{c}_{0.9500}$	$\tilde{c}_{0.9750}$	$\tilde{c}_{0.9900}$	$\tilde{c}_{0.9950}$
Constant	0.6342	0.6857	0.7381	0.8209	0.8540
First order	0.4631	0.4798	0.5107	0.5425	0.5538
Second order	0.4380	0.4479	0.4867	0.4976	0.4977

Table 1. The simulated $\tilde{c}_{(1-\alpha)}$, for $h(t,s) = ts$, $(t,s) \in \mathbf{I}$.

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