E-Jurnal Matematika Vol. 7 (2), Mei 2018, pp. 187-193 ISSN: 2303-1751
DOI: https://doi.org/10.24843/MTK.2018.v07.i02.p202

THE EXISTENCE AND UNIQUENESS OF COMPLETION OF COMPLEX VALUED METRIC SPACES

Dahliatul Hasanah[1§], Imam Supeno[2]

[1]Jurusan Matematika, Fakultas MIPA – Universitas Negeri Malang [Email: dahliatul.hasanah.fmipa@um.ac.id]
[2]Jurusan Matematika, Fakultas MIPA – Universitas Negeri Malang [Email: imam.supeno.fmipa@um.ac.id]
[§]*Corresponding Author*

ABSTRACT

Completion of a metric space has been investigated and proved its existence and uniqueness. While a complex valued metric space is considered as a generalization of a metric space, we study the completion properties of complex valued metric spaces. Using redefined definitions of isometry and denseness in complex valued metric spaces which correspond to the definitions in metric spaces, we prove the existence and uniqueness of completion of a complex valued metric space.

***Keywords**: complex valued metric spaces, completion, isometry, denseness*

1. INRODUCTION

Complex valued metric spaces are considered as a generalization of the classical metric spaces by redefining the metric. It was introduced by Azzam et.al (2011) to investigate the existence of common fixed points of a pair of mappings satisfying contractive type condition in the new spaces. Following the definition of complex valued metric spaces, many researchers started to investigate other fixed point theorems in the spaces such as in Sintunavarat & Kumam (2012), Sitthikul, et. Al. (2012), and Nashine, et. al. (2014).

A well-known property of metric spaces is the existence and uniqueness of completion of a metric space. In other words, every metric space has a completion. Completion of other type of metric spaces is also studied in recent years. For example, Ge & Lin (2015) investigate the existence of completion of partial metric spaces and Dung (2017) provides an example of completion of partial metric spaces as an answer to Ge & Lin's question about denseness property. On the other hand, Cevik & Okezen (2016) study the completion of multiplicative metric spaces. Completion of quasi-pseudometric spaces is investigated by Andrikopoulos (2013). It is still an open problem whether a complex valued metric space has a completion. To investigate the existence and uniqueness of completion, we introduce some notions in complex valued metric spaces such as isometry and denseness. The definitions of new notions correspond to the same notions in metric spaces.

2. PRELIMINARIES

In this section we give some preliminaries on complex valued metric spaces and some basic definitions and properties in the spaces.

Definition 2.1. (Azzam, et.al., 2011) *A partial order $\precsim$ on $\mathbb{C}$ is defined as follows.*

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \le Re(z_2) \text{ and } Im(z_1) \le Im(z_2).$$

From the definition, it follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

i. $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$
ii. $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$
iii. $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$
iv. $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$

The notation $z_1 \lnsim z_2$ is used only if the condition (i), (ii), or (iii) is satisfied. If the condition (iii) is satisfied, then it is written as $z_1 \prec z_2$.

The properties of the order $\precsim$ are given by

a) $0 \precsim z_1 \precnsim z_2$ implies $|z_1| < |z_2|$
b) $z_1 \precsim z_2$ and $z_2 \prec z_3$ implies $z_1 \prec z_3$.

Definition 2.2. (Azzam, et.al., 2011) *Let X be a nonempty set. Suppose that for all $x, y, z \in X$, the mapping $d: X \times X \to \mathbb{C}$ satisfies*

(C1) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(C2) $d(x, y) = d(y, x)$;
(C3) $d(x, y) \precsim d(x, z) + d(z, y)$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

The followings are given definition of interior and limit point in complex valued metric spaces and definition of open and closed sets in the spaces.

A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X: d(x, y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset.$$

A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

Convergence of a sequence and a Cauchy sequence in complex valued metric spaces are defined as follows.

Let (x_n) be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x, x_n) \prec c$, then (x_n) is said to be convergent. Sequence (x_n) converges to x and x is said to be the limit point of (x_n). This convergence is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then (x_n) is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued metric space.

The following are lemmas related to properties of sequences in complex valued metric spaces.

Lemma 2.3. (Azzam, et.al., 2011) *Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.4. (Azzam, et.al., 2011) *Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 2.5. *Let (X, d) be a complex valued metric space. Let (x_n) and (y_n) be sequences in X such that $(x_n) \to x$ and $(y_n) \to y$. Then $d(x_n, y_n) \to d(x, y)$.*

PROOF. By the property of complex valued metric, we have

$$d(x_n, y_n) \precsim d(x_n, x) + d(x, y) + d(y, y_n).$$

Taking the absolute value and limit of both sides, we get

$$\lim|d(x_n, y_n)| \le \lim|d(x_n, x)| + \lim|d(x, y)| + \lim|d(y, y_n)| = \lim|d(x, y)| = |d(x, y)|.$$

On the other hand,

$$d(x, y) \precsim d(x_n, x) + d(x_n, y_n) + d(y, y_n).$$

Taking the absolute values and limit of both sides, we have

$$|d(x, y)| \le \lim|d(x_n, y_n)|.$$

Hence we have $\lim|d(x_n, y_n)| = |d(x, y)|$ or we can say that $d(x_n, y_n) \to d(x, y)$. □

The followings are the definitions of isometry and denseness in a complex valued metric space by adopting the description of isometry and denseness for partial metric case in Ge & Lin (2014).

Definition 2.6. *Let (X, d) and (Y, ρ) be complex valued metric spaces. A mapping $f: X \to Y$ is called an isometry if $\rho(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.*

The following lemma states the relation between isometry and convergence of a sequence.

Lemma 2.7. *Let $f: X \to Y$ be an isometry, where (X, d) and (Y, ρ) are complex valued metric spaces. If (x_n) is a sequence in X which*

E-Jurnal Matematika Vol. 7 (2), Mei 2018, pp. 187-193 ISSN: 2303-1751
DOI: https://doi.org/10.24843/MTK.2018.v07.i02.p202

converges to x, then $(f(x_n))$ is a sequence in Y converging to $f(x)$.

PROOF. Suppose that (x_n) is in X and $x_n \to x$. We will show that $(f(x_n))$ converges to $f(x)$.

For any $0 \prec c \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. On the other hand, $\rho(f(x_n), f(x)) = d(x_n, x)$. Hence $\rho(f(x_n), f(x)) \prec c$. This means that $(f(x_n))$ converges to $f(x)$. □

Definition 2.8. *A subset M of a complex valued metric space X is said to be dense in X if $\bar{M} = X$.*

Form the definition above, taking an element of M would yield an element of M itself or a limit point of M. Thus this means that a subset M of a complex valued metric space X is called dense in X if for each element of$M$, there exist an element of X in every neighborhood of the element of M.

Other that denseness in Definition 2.8, we introduce the notion of sequentially dense in complex valued metric spaces as the same notion that was introduced by Ge and Lin (2015) in partial metric spaces.

Definition 2.9. *Let (X, d) be a complex valued metric space and M be a subspace of X. M is called sequentially dense in X if for any $x \in X$ there is a sequence in M converging to x.*

There is a equivalent relation between the notion of dense set and sequentially dense set in complex valued metric spaces as stated in the Lemma as follows.

Lemma 2.10. *Let X be a complex valued metric space. Let M be a subset of X. The set M is dense in X if and only if it is sequentially dense in X.*

PROOF. Let M be a dense subset of X and $x \in X$. Then x is in M or x is a limit point of M. Suppose that x is in X. Then the sequence $(x, x, x, \dots)$ in M converges to x. If x is not in X then x is a limit point of M.
Taking $r_1 = 1 + i\,, 0 \prec r_1$, there exists $x_1 \in M$ such that $x_1 \in B(x, r_1) \cap (M \setminus \{x\})$.

$r_2 = \frac{1}{2} + i\frac{1}{2}, 0 \prec r_2$, there exists $x_2 \in M$ such that $x_2 \in B(x, r_2) \cap (M \setminus \{x\})$ and $x_1 \neq x_2$.
Using similar fashion, for every $r_n = \frac{1}{n} + i\frac{1}{n}\,, \forall n \in \mathbb{N}$, there exists $x_n \in M$ such that $x_n \in B(x, r_n) \cap (M \setminus \{x\})$.
We have a sequence $(x_1, x_2, x_3, \dots)$ in M such that

$$d(x_n, x) \prec r_n = \frac{1}{n} + i\frac{1}{n}, \forall n \in \mathbb{N}.$$

We will show that $(x_n) \to x$.
For any $c = c_1 + i\,c_2, c_1, c_2 \in \mathbb{R}, 0 \prec c$, using the Archimedean property, there exist $n_1, n_2 \in \mathbb{N}$ such that $\frac{1}{c_1} < n_1$ and $\frac{1}{c_2} < n_2$. Hence $\frac{1}{n_1} + i\frac{1}{n_2} \prec c_1 + i\,c_2$.
Taking $n_0 = \max\{n_1, n_2\}$ so that for all $n > n_0$,

$$d(x_n, x) \prec r_n = \frac{1}{n} + i\frac{1}{n} \precsim \frac{1}{n_0} + i\frac{1}{n_0}$$
$$\precsim \frac{1}{n_1} + i\frac{1}{n_2} \prec c_1 + i\,c_2 = c.$$

So (x_n) converges to x. Thus for every x in X there exists a sequence in M which is converging to x. M is sequentially dense in X.
On the other hand, suppose that M is sequentially dense in X. We will show that every x in X is in $\bar{M}$.
If $x \in M$ then $x \in \bar{M}$. Suppose that $x \notin M$, there is (x_n) in M such that $(x_n) \to x$. We will show that x is a limit point of M. For any $0 \prec c \in \mathbb{C}$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. Take $x_{n_1} \in M$ where $n_1 > n_0$ then $x_{n_1} \neq x$ and $x_{n_1} \in B(x, c) \cap (M \setminus \{x\})$. So x is a limit point of M. Hence sequentially denseness implies denseness.

The following definition gives the basic concept of completion in the sense of complex valued metric spaces.

Definition 2.11. *Let (X, d) be a complex valued metric space. A complete complex valued metric space (X^*, d^*) is called a completion of (X, d) if there exists an isometry $f: X \to X^*$ such that $f(X)$ is dense in X^*.*

3. RESULTS AND DISCUSSION

In this section we give the theorems of existence and uniqueness of completion for complex valued metric spaces. Theorem 3.1 states the existence of the completion, while the uniqueness of the completion is mentioned in Theorem 3.2. To prove the existence of completion, some steps must be taken. The first step is to construct a complex valued metric spaces. The next step is to construct an isometry based on Definition 2.6 from X to some subset of the costructed complex valued metric space. The next step that follows is to prove that the subset is dense based on the Definition 2.9. The final step is to prove that the complex valued metric space is complete.

Theorem 3.1. *Every complex valued metric space has a completion.*

PROOF. Let (X, d) be a complex valued metric space. Define a set $\mathcal{C}$ as

$$\mathcal{C} = \{(x_n) : (x_n) \text{ is Cauchy sequence in } X\}$$

We define a relation $\sim$ on $\mathcal{C}$ as follows:
For (x_n) and (y_n) in $\mathcal{C}$, $(x_n) \sim (y_n) \Leftrightarrow \lim|d(x_n, y_n)| = 0$.
We will show that the relation $\sim$ is an equivalence relation.

(i) Reflexive
Note that for $(x_n) \in \mathcal{C}$, based on the property complex valued metric, $d(x_n, x_n) = 0, \forall n \in \mathbb{N}$. Hence $\lim|d(x_n, y_n)| = 0$. So $(x_n) \sim (x_n)$.

(ii) Symmetry
Suppose that (x_n), $(y_n) \in \mathcal{C}$ satisfy $(x_n) \sim (y_n)$. Based on the property of complex valued metric, $0 = \lim|d(x_n, y_n)| = \lim|d(y_n, x_n)|$. Thus we have $(y_n) \sim (x_n)$.

(iii) Transitive
Suppose that (x_n), $(y_n), (z_n) \in \mathcal{C}$ satisfy $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$. We have,

$$d(x_n, y_n) \precsim d(x_n, z_n) + d(z_n, y_n).$$

By the property of partial order $\precsim$,

$$|d(x_n, z_n)| \le |d(x_n, y_n) + d(y_n, z_n)| \le |d(x_n, y_n)| + |d(y_n, z_n)|.$$

Taking limit both sides yields

$$\lim|d(x_n, z_n)| \le \lim|d(x_n, y_n)| + \lim|d(y_n, z_n)| = 0 + 0 = 0.$$

Thus $\lim|d(x_n, z_n)| = 0$ and hence $(x_n) \sim (z_n)$.

So, the relation $\sim$ is an equivalence relation.

Let X^* be the set of all equivalence classes in $\mathcal{C}$ for the equivalence relation $\sim$, that is $X^* = \{[(x_n)] : (x_n) \in \mathcal{C}\}$. We write $(x'_n) \in [(x_n)]$ to mean that $(x'_n) \sim (x_n)$ and (x'_n) is a representative of the equivalence class $[(x_n)]$. We now define $d^* : X^* \times X^* \to \mathbb{C}$ as follows: for $[(x_n)], [(y_n)] \in X^*$,

$$d^*([(x_n)], [(y_n)]) = \lim d(x_n, y_n).$$

We will show that d^* is well-defined by showing that $\lim d(x_n, y_n)$ exists for arbitrary Cauchy sequences (x_n) and (y_n) in X and the limit is independent of the particular choice of the representatives of the equivalence classes.

(i) For Cauchy sequences (x_n) and (y_n) in X,

$$d(x_n, y_n) \precsim d(x_n, x_{n+m}) + d(x_{n+m}, y_{n+m}) + d(y_{n+m}, y_n)$$

$$d(x_n, y_n) - d(x_{n+m}, y_{n+m}) \precsim d(x_n, x_{n+m}) + d(y_{n+m}, y_n)$$

$$|d(x_n, y_n) - d(x_{n+m}, y_{n+m})| \precsim |d(x_n, x_{n+m})| + |d(y_{n+m}, y_n)|$$

By taking limit on both sides we have

$$\lim|d(x_n, y_n) - d(x_{n+m}, y_{n+m})| = \lim|d(x_n, x_{n+m})| + \lim|d(y_{n+m}, y_n)|.$$

From Lemma 2, we have
$\lim|d(x_{n+m}, x_n)| = \lim|d(y_{n+m}, y_n)| = 0$.
Hence, $\lim|d(x_n, y_n) - d(x_{n+m}, y_{n+m})| = 0$.
This means that $d(x_n, y_n)$ is Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is a complete then $d(x_n, y_n)$ converges in $\mathbb{C}$, thus $\lim d(x_n, y_n)$ does exist.

(ii) Let $(x_n), (x'_n), (y_n), (y'_n)$ be in $\mathcal{C}$ such that $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$. We will show that the limit is independent of the particular choice of the representatives of the equivalence classes by showing that $\lim d(x_n, y_n) = \lim d(x'_n, y'_n)$.
By the property of complex valued metric and the equivalence relation,

$$d(x_n, y_n) \precsim d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n)$$

 ISSN: 2303-1751
DOI: https://doi.org/10.24843/MTK.2018.v07.i02.p202

$$d(x_n, y_n) - d(x'_n, y'_n) \precsim d(x_n, x'_n) + d(y'_n, y_n)$$

$$\lim|d(x_n, y_n) - d(x'_n, y'_n)| \le \lim|d(x_n, x'_n)| + \lim|d(y'_n, y_n)| = 0.$$

This would imply $\lim[d(x_n, y_n) - d(x'_n, y'_n)] = 0$.

Since $d(x_n, y_n)$ is convergent sequence in $\mathbb{C}$, then $d(x'_n, y'_n)$ is also convergent. Thus $\lim d(x'_n, y'_n)$ does exist and

$$\lim d(x_n, y_n) = \lim d(x'_n, y'_n).$$

Consequently, $d^*: X^* \times X^* \to \mathbb{C}$ is well-defined.

We will show that d^* is complex valued metric on X^*.

(i) Let $[(x_n)], [(y_n)] \in X^*$. We have $d^*([(x_n)], [(y_n)]) = \lim d(x_n, y_n)$. Since $0 \prec d(x_n, y_n)$, then $0 \prec \lim d(x_n, y_n)$. Hence d^* satisfies the first axiom of complex valued metric.

(ii) Let $[(x_n)], [(y_n)] \in X^*$. Note that using the property of complex valued metric d we have $d^*([(x_n)], [(x_n)]) = \lim d(x_n, x_n) = 0$. On the other hand, if we have $0 = d^*([(x_n)], [(y_n)]) = \lim d(x_n, y_n)$, then this implies $\lim|d(x_n, y_n)| = 0$. This would mean that $(x_n) \sim (y_n)$. Hence $[(x_n)] = [(y_n)]$.

(iii) For $[(x_n)], [(y_n)] \in X^*$,
$d^*([(x_n)], [(y_n)]) = \lim d(x_n, y_n) = \lim d(y_n, x_n) = d^*([(y_n)], [(x_n)])$.

(iv) $d^*([(x_n)], [(z_n)]) = \lim d(x_n, z_n) \precsim \lim d(x_n, y_n) + \lim d(y_n, z_n) = d^*([(x_n)], [(y_n)]) + d^*([(y_n)], [(z_n)])$.

The property of triangle inequality completes the proof that d^* is complex valued metric on X^*.

The next step is to construct an isometry from X to X^* based on Definition 2.6. For each $x \in X$, let x^* be the equivalence class of the constant sequence $(x, x, x, \dots)$, that is $x^* = [(x, x, x, \dots)] \in X^*$. We define $f: X \to X^*$ by $f(x) = x^*$. We will show that f is an isometry.

For $x, y \in X$,

$$d^*(f(x), f(y)) = d^*(x^*, y^*) = \lim d(x, y) = d(x, y).$$

Based on Definition 2.6, $f: X \to X^*$ is an isometry.

We will show that $f(X)$ is dense in X^*. We consider any $x^* \in X^*$. Let $(x_n) \in x^*$. Thus (x_n) is Cauchy sequence in X, that is $\forall 0 \prec c \in \mathbb{C}, \exists n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) \prec c$ for all $n > n_0$. Let $(x_{n_0}, x_{n_0}, x_{n_0}, \dots) \in [(x_{n_0})]$. Then we have $[(x_{n_0})] \in f(X)$. From the definition of d^* we have

$$d^*([(x_{n_0})], x^*) = d^*([(x_{n_0})], [(x_n)]) = \lim d(x_{n_0}, x_n).$$

Since $d(x_{n_0}, x_n) \prec c, \forall n > n_0$, then we have $\lim d(x_{n_0}, x_n) \prec c$.

So we get $d^*([(x_{n_0})], x^*) = \lim d(x_{n_0}, x_n) \prec c, \forall 0 \prec c \in \mathbb{C}$. This means that every neighborhood of arbitrary $x^* \in X^*$ contains an element of $f(X)$. Thus by Definition 2.8, $f(X)$ is dense in X^*.

The last step is to show that X^* is a complete complex valued metric space, that is every Cauchy sequence in X^* is convergent in X^*. Let (x_n^*) be any Cauchy sequence in X^*. Since $f(X)$ is dense in X^*, then for every x_n^*, there is $y_n^* \in f(X)$ such that

$$d^*(x_n^*, y_n^*) \prec \frac{1}{n} + i\frac{1}{n}.$$

This implies $\lim|d^*(x_n^*, y_n^*)| = 0$.

We will show that (y_n^*) is also a Cauchy sequence in X^*. By the triangle inequality of complex valued metric, we get

$$d^*(y_n^*, y_{n+m}^*) \precsim d^*(y_n^*, x_n^*) + d^*(x_n^*, x_{n+m}^*) + d^*(x_{n+m}^*, y_{n+m}^*).$$

Taking modulus of each side,

$$|d^*(y_n^*, y_{n+m}^*)| \le |d^*(y_n^*, x_n^*)| + |d^*(x_n^*, x_{n+m}^*)| + |d^*(x_{n+m}^*, y_{n+m}^*)|.$$

Then,

$$\lim|d^*(y_n^*, y_{n+m}^*)| = 0.$$

Hence (y_n^*) is Cauchy sequence in $f(X)$.

Suppose that $y_n \in X$ such that $f(y_n) = y_n^*, \forall n$. We will prove that (y_n) is a Cauchy sequence in X. Note that

$$d^*(y_n^*, y_{n+m}^*) = d^*(f(y_n), f(y_{n+m})) = d(y_n, y_{n+m}).$$

Taking modulus and limit on both sides, we have

$$\lim|d(y_n, y_{n+m})| = \lim|d^*(y_n^*, y_{n+m}^*)| = 0.$$

Hence, (y_n) is Cauchy sequence in X.

Suppose that $x^* \in X^*$ is the equivalence class in X^* in which (y_n) belongs. We will show that (x_n^*) is converging to x^*. By the triangle inequality,

$$d^*(x_n^*, x^*) \precsim d^*(x_n^*, y_n^*) + d^*(y_n^*, x^*).$$

For fixed n, the constant sequence $(y_n, y_n, y_n, \dots)$ belongs to y_n^* and the sequence (y_m) is in x^*. So $d^*(y_n^*, x^*) = \lim(y_n, y_m)$. Thus we have

$$d^*(x_n^*, x^*) \precsim d^*(x_n^*, y_n^*) + \lim(y_n, y_m).$$

Since we have $\lim|d^*(x_n^*, y_n^*)| = 0$, taking modulus and limit on both sides yields

$$\lim d^*(x_n^*, x^*) = 0.$$

This implies (x_n^*) converges to x^*. Hence (X^*, d^*) is complete complex valued metric space.

This completes the proof that every complex valued metric space has a completion. □

Theorem 3.2. *The completion of a complex valued metric space* (X, d) *is unique with respect to isometry under denseness. More precisely, if* (X^*, d^*) *and* $(\tilde{X}, \tilde{d})$ *are two completions of* (X, d), *then there is a unique isometry from* X^* *to* $\tilde{X}$.

PROOF. Suppose that (X^*, d^*) and $(\tilde{X}, \tilde{d})$ are two completions of (X, d). Let $f: X \to X^*$ and $g: X \to \tilde{X}$ be the isometries by which X^* and $\tilde{X}$ are completions of X. Note that the inverse of f, that is f^{-1}, is also an isometry. Consider the mapping $g \circ f^{-1}: f(X) \to g(X)$ where $f(X) \subseteq X^*$ and $g(X) \subseteq \tilde{X}$. Now we will define a mapping $h: X^* \to \tilde{X}$ such that the mapping is an isometry. Define the mapping h as follows.

- If $x^* \in X^*$ and $x^* \in f(X)$, then $h(x^*) = (g \circ f^{-1})(x^*)$
- If $x^* \in X^*$ and $x^* \notin f(X)$, we have $f(X)$ is dense in X^*. Based on Lemma 2.10, $f(X)$ is sequentially dense in X^*. Thus there exists a sequence $(x_n^*) \subseteq f(X)$ such that $(x_n^*) \to x^*$. Since $(x_n^*) \subseteq f(X)$, then there is $x_n \in X$ such that $f(x_n) = x_n^*$ for all $n \in \mathbb{N}$. Since (x_n^*) is Cauchy sequence in X^* and f is an isometry then (x_n) is also a Cauchy sequence in X. By the isometry $g: X \to \tilde{X}$, the sequence $(g(x_n)) = (\widetilde{x_n})$ is also a Cauchy sequence in $\tilde{X}$. By the completeness of $\tilde{X}$, $(\widetilde{x_n})$ is converging to some $\tilde{x}$ in $\tilde{X}$. Put $h(x^*) = \tilde{x}$.

We will show that the mapping $h: X^* \to \tilde{X}$ is well-defined.

For $x^*, y^* \in f(X) \subseteq X^*$ and $x^* = y^*$, $h(x^*) = (g \circ f^{-1})(x^*) = (g \circ f^{-1})(y^*) = h(y^*)$ since f and g are isometries. Now we consider $x^* \in X^*$ and $x^* \notin f(X)$. The set $f(X)$ is sequentially dense in X^*, hence we suppose that (x_n^*) and (y_n^*) are sequences in $f(X)$ which are converging to x^*. For all n, we have x_n^* and y_n^* are in $f(X)$ so $h(x_n^*)$ and $h(y_n^*)$ are in $\tilde{X}$. By the isometries of f and g, $(h(x_n^*))$ and $(h(y_n^*))$ are Cauchy sequences and thus are convergent in $\tilde{X}$. Note that

$$\begin{aligned}
&\tilde{d}(h(x_n^*), h(y_n^*)) \\
&= \tilde{d}((g \circ f^{-1})(x_n^*), (g \circ f^{-1})(y_n^*)) \\
&= \tilde{d}\left(g(f^{-1}(x_n^*)), g(f^{-1}(y_n^*))\right) \\
&= d(f^{-1}(x_n^*), f^{-1}(y_n^*)) \\
&= d^*\left(f(f^{-1}(x_n^*)), f(f^{-1}(y_n^*))\right) \\
&= d^*(x_n^*, y_n^*).
\end{aligned}$$

Since (x_n^*) and (y_n^*) converge to x^*, so we have

$$\begin{aligned}
\left|\tilde{d}(h(x_n^*), h(y_n^*))\right| &= |d^*(x_n^*, y_n^*)| \\
&\leq |d^*(x_n^*, x^*)| + |d^*(x^*, y_n^*)| \\
&\to 0.
\end{aligned}$$

Suppose that $(h(x_n^*)) \to x$ and $(h(y_n^*)) \to y$, then we get

$$\begin{aligned}
|d^*(x, y)| \leq &\left|d^*(x, h(x_n^*))\right| \\
&+ \left|d^*(h(x_n^*), h(y_n^*))\right| \\
&+ |d^*(h(y_n^*), y)| = 0.
\end{aligned}$$

By the properties of complex valued metric, $x = y$. So $(h(x_n^*))$ and $(h(y_n^*))$ are converging to the same limit. Thus the mapping $h: X^* \to \tilde{X}$ is well-defined.

The next step is to show that the mapping is an isometry. Let $x^*, y^* \in X^*$. Then there are sequences (x_n^*) and (y_n^*) in $f(X)$ which converge to x^* and y^* respectively. By Lemma 2.7, $(h(x_n^*)) \to h(x^*)$ and $(h(y_n^*)) \to h(y^*)$. This means

$$\begin{aligned}
&\tilde{d}(h(x^*), h(y^*)) \\
&= \lim \tilde{d}(h(x_n^*), h(y_n^*)) \\
&= \lim \tilde{d}((g \circ f^{-1})(x_n^*), (g \circ f^{-1})(y_n^*)) \\
&= \lim d^*(x_n^*, y_n^*) \\
&= d^*(x^*, y^*).
\end{aligned}$$

So the mapping $h: X^* \to \tilde{X}$ is an isometry.

To prove the uniqueness of the isometry, we will show that if there exists other isometry from X^* to $\tilde{X}$, then they are the same. Let $j: X^* \to \tilde{X}$ be an isometry. We will show that $h = j$.

Let $x^* \in X^*$, then there is a sequence $(x_n^*) \subseteq f(X)$ which is converging to x^*. The mapping j is an isometry, then

$$\left|\tilde{d}(j(x_n^*), j(x^*))\right| = |d^*(x_n^*, x^*)| \to 0.$$

So $(j(x_n^*))$ converges to $j(x^*)$. On the other hand, $(j(x_n^*))$ is also converging to $h(x^*)$. By the uniqueness of limit of sequence, we have $j(x^*) = h(x^*)$.

We complete the proof that the completion of a complex-valued metric space is unique. □

4. CONCLUSION AND SUGGESTION

In this paper, we propose the definitions of isometry and denseness in complex valued metric spaces based on those in metric spaces. The definitions are used to prove the existence and uniqueness of complex valued metric spaces. Hence complex valued metric spaces follow metric spaces in term of the property of having completion for each metric space.

REFERENCES

Andrikopoulos, A. 2013. A solution to the completion problem for quasi-pseudometric spaces. *International Journal of Mathematics and Mathematical Sciences*. doi: 10.1155/2013/278381

Azzam, A, Fisher, B, Khan, M. 2011. Common fixed point theorems in complex valued metric spaces. *Numer. Funct. Anal. Optim* 32(3), p.243-253.

Cevik, C., Okezen, C.C. 2016. Completion of multiplicative metric spaces. *Gazi University Journal of Science* 29(3), p.663-666

Dung, N.V. 2017. On the completion of partial metric spaces. *Quaestiones Mathematicae* 40(5), p.589-597. doi: 10.2989/16073606.2017.1303004

Ge, X., Lin, S. 2015. Completions of partial metric spaces. *Topol. and its applications* 182, p.16-23. doi:10.1016/j.topol.2014.12.013.

Nashine, H.K., Imdad, M., Hasan, M. 2014. Common fixed point theorems under rational contractions in complex valued metric spaces. *J. Nonlinear Sci. Appl*. 7, p.42-50.

Sintunavarat, W., Kumam, P. 2012. Generalised common fixed point theorems in complex valued metric spaces and applications. *J. Ineq. Appl.* doi:10.1186/1029-242X-2012-84

Sitthikul, K., Saejung, S. 2012. Some fixed point theorems in complex valued metric spaces. *Fixed Point Theo. Appl*. p.189.doi:10.1186/1687-1812-2012-189