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PERIODIC SOLUTION OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH NONLINEAR BOUNDARY CONDITION

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ABSTRACT

In this paper we study the existence and approximation of the periodic solution of nonlinear integro-differential equation with nonlinear boundary condition by assuming the function $f(t,x,y)$ is a measurable at t and bounded by Lebesgue integrable function which has the weaker conditions. The numerical –analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by Samoilenko.

Keywords: *Existence, uniqueness and stability of a periodic solution, Lebesgue integrable integro-differential equation, boundary condition.*

1. INTRODUCTION

A boundary value problem consists of a integro- differential equation on a given interval and an explicit condition that the solution must satisfy at one or several points. The simplest instance of such explicit conditions is when they are all specified at one initial point. The solution of differential equations may be generally specified at more than one point. Often there are two points, which correspond physically to the boundaries of some region, so that it is a two–point boundary value problem (Putertnka,1991; Ronto, 2000).

The theory of integro-differential equations has been of great interest for many years. It plays an important role in different subjects, such as physics, biology, chemistry, etc, and the study of periodic solutions for non-linear system of differential equations with boundary conditions and boundary integral conditions is a very important branch in the differential equation theory (Robert, 1984; Mitropolsky,1979: Sahla 2021).

Many results about the existence and approximation of periodic solutions for system of non–linear differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko (1976) which had been later applied in many studies (Butris, 1994; Perestyuk, 1974; Shslapk, 1980).

The so-called numerical–analytic method for investigating a periodic solution, is widely used for studying solvability of non-linear boundary value problems and constructing approximate solutions for finding harmonic oscillations arising in various systems described by integro –differential equations, and differential equations with boundary conditions.

Samoilenko (1976) has used the numerical-analytic methods of periodic solutions for ordinary differential equation which has the form

$$\frac{dx}{dt} = f(t,x)$$

where $x \in D$,
D is closed and bounded subset of R^n,
the vector function $f(t,x)$ is defined on the domain:
$(t,x) \in R^1 \times D = (-\infty,\infty) \times D$,
which is continuous in t and x and periodic in t of period T.

Lemma 1 (Samoilenko 1976). Let f(t) be a vector function which is defined in the interval $0 \le t \le T$, then:

$$\left\|\int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds\right\| \le \alpha(t)M ,$$

where $M = \max_{t\in[0,T]} |f(t)|$ and $\alpha(t) = 2t(1 - \frac{t}{T})$

In this work, we investigate the existence and approximation of periodic solutions for non-linear system of integro-differential equation with nonlinear boundary condition which has the form:

$$\frac{dx}{dt} = f\left(t, x, \int_{a(t)}^{b(t)} g(s, x(s))ds\right) \qquad (1)$$

$$\langle x(0), x(T) \rangle = c \ , \ \ c \in R^1 = (-\infty, \infty) \qquad (2)$$

where $\langle \ . \ , \ . \ \rangle$ is a scalar product of vectors $x(0)$ and $x(T)$, x_0, $f-$ are points of the n-dimensional Euclidean space R^n.

Assume that the functions $f(t,x,y) = (f_1(t,x,y), f_2(t,x,y), \dots, f_n(t,x,y))$ and $g(t,x) = (g_1(t,x), g_2(t,x), \dots, g_n(t,x))$ are define and continuous in the domain

$$(t,x,y) \in R^1 \times D \times D_1 = (-\infty, \infty) \times D \times D_1 \qquad (3)$$

and periodic in t of period T, where D is a closed bounded domain in R^n and D_1 is bounded on the same domain, $R^n$is Euclidean spaces .

Furthermore, the vector functions $f(t,x,y)$ and $g(t,x)$ are satisfy the following inequalities.

$$\|f(t,x,y)\| \le \|M(t)\|, \|g(t,x)\| \le \|N(t)\| \quad (4)$$

$$\left.\begin{aligned} &\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le \\ &\|K_1(t)\|\|x_1 - x_2\| + \|K_2(t)\|\|y_1 - y_2\| \\ &\|g(t,x_1) - g(t,x_2)\| \le \|L(t)\|\|x_1 - x_2\| \end{aligned}\right\} (5)$$

for all $t \in R^1$, $x, x_1, x_2 \in D$ and $y, y_1, y_2 \in D_1$ where $M(t), N(t), K_1(t), K_2(t), L(t)$ are Lebesgue integrable functions,

Also $H = \|b(t) - a(t)\|$ are positive constant, $\|.\| = \max_{t\in[0,T]} |.|$.

Definition1 . A function f is defined on a set $E \subseteq R^1$ is said to be continuous a point x in E if $\epsilon > 0$ is given , there is a positive number δ , such that for all y in E with $\|x - y\| < \delta$ we have $\|f(x) - f(y)\| < \epsilon$.

Definition2. let f be a continuous function define on the domain: $G = \{(t,x) : a \le t \le b, c \le x \le d\}$ then f is said to satisfy a Lipchitz condition in the variable x on G ,provided that a constant if for all $K > 0$ exists with the property that

$$\|f(t,x_1) - f(t,x_2)\| \le K\,\|x_1 - x_2\|$$

For all $(t,x_1), (t,x_2) \in G$ the constant K is called a Lipschitz constant for f .

Beside (1) , we can also consider the following system:

$$\frac{dx}{dt} = f\left(t, x, \int_{a(t)}^{b(t)} g(s, x(s))ds\right) - \Delta \qquad (6)$$

with nonlinear condition (2) , where $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_n)$ is a parameter.

Definition3. The value of the parameter Δ at which the solution of the system (6) taking for $t = 0$ the value $x = x_0$ is periodic in t of period T and such a Δ is a unique will be called a $\Delta-$ constant at the point (t, x_0) with respect to the system (6) .

Definition4 . A linear space E with a norm defined on it is called a normed space.

Definition5 . let $(E, \|.\|)$ be a norm space, If T map into itself we say that T is a contraction mapping on E if there exists $\partial \in R$ with $0 < \partial < 1$ such that

$$\|Tx(t) - Ty(t)\| \le \partial\|x(t) - y(t)\|,$$
$$x(t), y(t) \in E .$$

Definition6 .A solution $x(t)$ is said to be stable if for each $\varepsilon > 0$, there exist a $\delta > 0$ such that any solution $\bar{x}(t)$ which satisfies $\|\bar{x}(t_0) - x(t_0)\| < \delta$ for some t_0, also satisfies $\|\bar{x}(t) - x(t)\| < \varepsilon$ for all $t \ge t_0$.

Definition7. Let f be a function on a set $E \subseteq R^1$. We say that f is Lebesgue measurable on E if for every $\propto \in R^1$, the set $\{t; t \in E, f(t) > \propto\}$ is measurable.

Definition 8 . Let f be Lebesgue measurable function defined on $E \subseteq R^1$ .Let $L(E)$ be the set of all measurable functions defined on E such that $\int_E \|f(t)\|dx < \infty$. The set $L(E)$ is called Lebesgue integrable functions.

Theorem 1. (Banach fixed point theorem) let E be a Banach space , if T is a contraction mapping on E then T has one and only one fixed point in E. (For the definitions see Rama (1984)**).**

We consider a sequence of a vector functions defined by the recurrence relation.

$$x_{m+1}(t,x_0) = x_0 + \int_0^t [f\left(s, x_m(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_m(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle\ x_0\ , \left(f(s, x_m(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_m(\tau,x_0))d\tau\right)\rangle ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \quad (7)$$

with ,

$$x_0(t,x_0) = x_0 + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \ ,$$

$m = 0, 1, 2, \ . \ . \ .$

at $x_0 \in D_f$, $\langle x_0, e\rangle \neq \emptyset$, $e = (1, 1, \ . \ . \ . \ , 1)$ unit vector .

Let $y_m(t,x_0) = \int_{a(t)}^{b(t)} g(s, x_m(s))ds$, $m = 0, 1, 2, \ . \ . \ .$ (8)

he non-empty sets are defined as follows:

$$\left.\begin{aligned} D_f &= D - \frac{T}{2}\|M(t)\| \\ D_{1f} &= D_1 - \frac{T}{2}\|M(t)\|\ \|L(t)\|H \end{aligned}\right\} \quad (9)$$

Furthermore , we suppose that the greatest eigenvalue$\lambda_{max}$of the matrix ,

$$\Lambda = \frac{T}{2}(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H) \ ,$$

does not exceed unity , i.e

$$\lambda_{max}(\Lambda) < 1 \ . \quad (10)$$

By using the Lemma 3.1 (Samoilenko, 1976) , we can state and prove the following Lemma

Lemma 1 . let $f(t,x_0)$ be a vector function which is defined in the interval $0 \le t \le T$, then:

$$\left|\int_0^t [f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle\ x_0\ , f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right)\rangle\ ds]ds\right| \le \|M(t)\|\alpha(t)$$

where $\|M(t)\| = \max_{t\in[0,T]} |f(t,x_0)|$ and

$\alpha(t) = 2t(1 - \frac{t}{T})$

let $\langle x_0, e\rangle = \sum_{i=1}^n x_{0i}$ and

$$\langle x_0, f(s,x_0)\rangle = \sum_{i=1}^n x_{0i}\, f_i(s,x_0)$$

$$\left|\int_0^t [f(s,x_0,y_0) - \frac{e}{T\sum_{i=1}^n x_{0i}}\int_0^T \sum_{i=1}^n x_{0i}\ f_i(s,x_0,y_0)ds]ds\right|$$

$$= \left|\int_0^t [f(s,x_0,y_0) - \frac{e}{T\sum_{i=1}^n x_{0i}}\sum_{i=1}^n x_{0i}\int_0^T f_i(s,x_0,y_0)ds]ds\right|$$

$$= \left|\int_0^t [f(s,x_0,y_0) - \frac{1}{T}\int_0^T f(s,x_0,y_0)ds]ds\right| \le \int_0^t |f(s,x_0,y_0)|ds - \frac{t}{T}\int_0^t |f(s,x_0,y_0)|ds + \frac{t}{T}\int_t^T |f(s,x_0\ ,y_0)|\ ds$$

$$\le \left(1 - \frac{t}{T}\right)t|f(t,x_0,y_0)| + \frac{t}{T}(T - t)|f(t,x_0,y_0)| \le \left(1 - \frac{t}{T}\right)t\|M(t)\| + \frac{t}{T}(T - t)\|M(t)\| \le 2t\left(1 - \frac{t}{T}\right)\|M(t)\|$$

$$\left|\ \int_0^t [f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0\ , f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right)\rangle\ ds]ds\ \right| \le \|M(t)\|\alpha(t)$$

2. APPROXIMATION OF A PERIODIC SOLUTION OF (1) , (2).

The investigation of periodic approximation solution of the system (1), (2) is formulated by the following theorem:

Theorem 2. if the system (1), (2) satisfies the inequalities (4), (5) and(6) with assumptions (9), (10), has periodic solution $x = x(t,x_0)$ passing through the point $(0,x_0)$, $x_0 \in D_f$, then the sequence of functions is:

$$x_m(t,x_0) = x_0 + \int_0^t [f\left(s, x_{m-1}(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_{m-1}(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle\ x_0, f\left(s, x_{m-1}(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_{m-1}(\tau,x_0))d\tau\right)\ \rangle\ ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \quad (11)$$

with $x_0(t,x_0) = x_0 + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle]$, $m = 1, 2, \ . \ . \ .$

is periodic in t of period , and uniformly convergent $m \to \infty$ in the domain:

$$(t,x_0) \in R^1 \times D_f = (-\infty,\infty) \times D_f \quad (12)$$

to the limit function $x^0(t,x_0)$ defined in the domain (3) which is periodic in t of period T and satisfying the system of integral equations:

$$x(t,x_0) = x_0 + \int_0^t [f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) \rangle ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \quad (13)$$

and it is a unique solution of the system (1.1), (1,2) provided that :

$$\|x^0(t,x_0) - x_0\| \le \|M(t)\|\,\alpha(t) \quad (14)$$

and

$$\|x^0(t,x_0) - x_m(t,x_0)\| \le \Lambda^m (E-\Lambda)^{-1}\|M(t)\|\,\alpha(t) \quad (15)$$

for all $m \ge 1$ and $t \in R^1$, where E is the identity matrix and

$$\alpha(t) = 2t\left(1 - \frac{t}{T}\right).$$

Proof. Consider the sequence of functions $x_1(t,x_0)$, $x_2(t,x_0)$, ... , $x_m(t,x_0)$, ... , defined by the recurrence relation (7) Each of these functions of sequence is defined and continuous in the domain (3) and periodic in t of period T .

Now, by the Lemma (1.2.2) , and using (1.12) , when $m = 1$, we get :

$$\|x_1(t,x_0) - x_0\| = \Big\| x_0 + \int_0^t \Big[f\left(s, x_0(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_0(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, x_0(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_0(\tau,x_0))d\tau\right)\rangle ds\Big] ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] - x_0 - \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle]\Big\|$$

$$= \Big\|\int_0^t [f\left(s, x_0(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_0(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, x_0(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x_0(\tau,x_0))d\tau\right)\rangle ds] ds\Big\|$$

$$\le \alpha(t)\|M(t)\|$$

and hence,

$$\|x_1(t,x_0) - x_0\| = \frac{T}{2}\|M(t)\|, \left(\alpha(t) \le \frac{T}{2}\right) \quad (16)$$

i.e $x_1(t,x_0) \in D$, for all $t \in R^1$, $x_0 \in D_f$

Also from (16) , we have:

$$\|y_1(t,x_0) - y_0(t,x_0)\| = \left\|\int_{a(t)}^{b(t)} g(s, x_1(s))ds - \int_{a(t)}^{b(t)} g(s, x_0(s))ds\right\|$$

$$\|y_1(t,x_0) - y_0(t,x_0)\| \le \|L(t)\|\|x_1(t,x_0) - x_0(t,x_0)\|H \quad (17)$$

For all $x_0 \in D_f$ and $y_0(t,x_0) = \int_{a(t)}^{b(t)} g(s, x(s))ds \in D_{1f}$.

i.e $y_1(t,x_0) \in D_1$,when $x_0 \in D_f$

Thus by induction ,we have :

$$\|x_m(t,x_0) - x_0(t,x_0)\| \le \|M(t)\|\,\alpha(t) \le \frac{T}{2}\|M(t)\| \quad (18)$$

for all $t \in R^1$, $x_0 \in D_f$ i.e

$x_m(t,x_0) \in D$ For all $t \in R^1$, $x_0 \in D_f$.

Now , from (18) ,we get :

$$\|y_m(t,x_0) - y_0(t,x_0)\| \le \|L(t)\|H\|x_m(t,x_0) - x_0(t,x_0)\| \quad (19)$$

$y_m(t,x_0) \in D_1$ For all $t \in R^1, x_0 \in D_f$

We claim that the sequence of functions (7) is uniformly convergent on the domain (12) .

by the Lemma 2.1 , and putting $m = 2$ in (7) , we have

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le \Big[\int_0^t \Big[(K_1(s)\|x_1(s,x_0) - x_0(s,x_0)\| + K_2(s)\|y_1(s,x_0) - y_0(s,x_0)\| - \frac{e}{T}\int_0^T K_1(s)\|x_1(s,x_0) - x_0(s,x_0)\| + K_2(s)\|y_1(s,x_0) - y_0(s,x_0)\|ds\Big] ds\Big]$$

$$\le \Big[\left(1 - \frac{t}{T}\right)t(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\|x_1(s,x_0) - x_0(s,x_0)\| + \frac{t}{T}(T-t)(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\|x_1(s,x_0) - x_0(s,x_0)\|\Big]$$

$$\le \Big[\left(t\left(1 - \frac{t}{T}\right) + \frac{t}{T}(T-t)\right)(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\|x_1(s,x_0) - x_0(s,x_0)\|\Big]$$

$$\le (\|K_1(t)\| + \|K_2(t)\|\|L(t)\|)\alpha(t)\|x_1(s,x_0) - x_0(s,x_0)\|$$

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le (\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\alpha(t)\|x_1(s,x_0) - x_0(s,x_0)\|$$

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le \Lambda\|x_1(t,x_0) - x_0(t,x_0)\|$$

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le \Lambda\|M(t)\|\alpha(t) \quad (20)$$

Suppose that the following inequality is true :

$$\|x_m(t,x_0) - x_{m-1}(t,x_0)\| \le \Lambda^{m-1}\|M(t)\|\alpha(t) \quad (21)$$

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We shall prove the following inequality

$$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \le \Lambda^m \|M(t)\|\alpha(t) \quad (22)$$

By using the inequalities (20) and (21), we get

$$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \le (\|K_1(t)\| + \|K_2(t)\|\|L(t)\|) \left[\left(1-\frac{t}{T}\right)\int_0^t \|x_m(s,x_0) - x_{m-1}(s,x_0)\|\, ds + \frac{t}{T}\int_t^T \|x_m(s,x_0) - x_{m-1}(s,x_0)\|\, ds\right]$$

So that:

$$\|x_{m+1}(t,x_0) - x_m(t,x_0)\| \le \Lambda^m \|M(t)\|\alpha(t)$$

Also is true $x_0 \in D_f$ and $m \ge 1$

From (21) and (22) , the following inequality:

$$\|x_{m+k}(t,x_0) - x_m(t,x_0)\| \le \|x_{m+k}(t,x_0) - x_{m+k-1}(t,x_0)\| + \|x_{m+k-1}(t,x_0) - x_{m+k-2}(t,x_0)\| + \ldots + \|x_{m+1}(t,x_0) - x_m(t,x_0)\|$$

$$\le \Lambda^{m+k-1}\|x_1(t,x_0) - x_0(t,x_0)\| + \Lambda^{m+k-2}\|x_1(t,x_0) - x_0(t,x_0)\| + \ldots + \Lambda^m\|x_1(t,x_0) - x_0(t,x_0)\|$$

$$\le \Lambda^m[1 + \Lambda + \Lambda^2 + \ldots + \Lambda^{k-2} + \Lambda^{k-1}]\|x_1(t,x_0) - x_0(t,x_0)\|$$

$$\|x_{m+k}(t,x_0) - x_m(t,x_0)\| \le \sum_{i=0}^{k-1}\Lambda^{m+i}\|x_1(t,x_0) - x_0(t,x_0)\| \le \sum_{i=0}^{k-1}\Lambda^{m+i}\|M(t)\|\alpha(t) \quad (23)$$

is hold for all $k > 1$ and $x_0 \in D_f$

But the Egin values of the matrix Λ are assumed to lie within the circle of a unit radius, which implies that:

$$\sum_{i=0}^{k-1}\Lambda^{m+i} \le \Lambda^m \sum_{i=0}^{k-1}\Lambda^i = \Lambda^m(E-\Lambda)^{-1} \quad (24)$$

and

$$\lim_{m\to\infty}\Lambda^m = 0 \quad (25)$$

Relation (23) and (25) prove the uniform convergence of the sequence of functions (7) in the domain (12) as $m \to \infty$

$$\lim_{m\to\infty} x_m(t,x_0) = x^0(t,x_0) \quad (26)$$

Since the sequence of functions (7) is periodic in t of period T, then the limiting function $x^0(t,x_0)$ is also periodic in t of period T.

Moreover, by the lemma 1 and the inequality (23) ,the inequality (14) and (15) are satisfied for all $m \ge 0$.

Using relation (26) and proceeding in (7) to the limit, when $m \to \infty$, this shows us that the limiting function $x^0(t,x_0)$ is the periodic solution of the integral equation (13) .

3. UNIQUENESS OF A PERIODIC SOLUTION OF (1), (2).

The study of periodic uniqueness solution of the system (1), (2) is formulated by the following theorem.

Theorem 3. A statement similar of all inequalities and conditions of the theorem 2. Then the system (1) with (2) is a unique periodic solutions on the domain(3).

We show that $x(t,x_0)$ is a unique solution of (1) , (2) . Assume that $x^*(t,x_0)$ is another solution of (1) , (2), i.e

$$x^*(t,x_0) = x_0 + \int_0^t [f\left(s, x^*(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x^*(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, x^*(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x^*(\tau,x_0))d\tau\right)\rangle ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \quad (27)$$

Now , we prove that $x(t,x_0) = x^*(t,x_0)$ for all $x_0 \in D_f$ and to do this , we need to prove the following inequality :

$$\|x^*(t,x_0) - x_m(t,x_0)\| \le \Lambda^m(E-\Lambda)^{-1}\|M^*\|\alpha(t) \quad (28)$$

Where

$$\|M^*\| = \max_{t\in[0,T]}\left|f\left(s, x^*(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x^*(\tau,x_0))d\tau\right)\right|$$

Suppose that (1.29) is true for $= k$, i.e

$$\|x^*(t,x_0) - x_k(t,x_0)\| \le \Lambda^k(E-\Lambda)^{-1}\|M^*\|\alpha(t) \quad (29)$$

Then,

$$\|x^*(t,x_0) - x_{k+1}(t,x_0)\| \le \left[\frac{T}{2}(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\right]\left[\left(1-\frac{t}{T}\right)\int_0^t \|x^*(s,x_0) - x_k(s,x_0)\|\, ds + \frac{t}{T}\int_t^T \|x^*(s,x_0) - x_k(s,x_0)\|\, ds\right] \le \Lambda^{k+1}(E-\Lambda)^{-1}M^*\alpha(t) \quad (30)$$

By induction, inequality (28) is true for $m = 0,1,2, \ldots$

Thus from (26) and (28), we have :

$$\lim_{m\to\infty}\|x^*(t,x_0) - x_m(t,x_0)\| \le (E - \Lambda)^{-1}\|M^*\|\alpha(t)\lim_{m\to\infty}\Lambda^m \quad (31)$$

From the condition(10) in (31) , we get:

$$\lim_{m\to\infty}\|x^*(t,x_0) - x_m(t,x_0)\| = 0$$

and hence ,

$$\lim_{m\to\infty} x_m(t,x_0) = x^*(t,x_0) \ .$$

By the inequality (23), we get $x(t,x_0) = x^*(t,x_0)$, i.e $x(t,x_0)$ is a unique solution of (1) , (2) .

4. EXISTENCE OF A PERIODIC SOLUTION (1)(2)

The problem of existence of periodic solution of period T of the system (1) , (2) is uniquely connected with the existence of zero of the function $\Delta(0,x_0)$ which has the form:

$$\Delta(t,x_0) = \frac{1}{T\langle x_0,e\rangle}[\langle x_0\,,x_0\rangle + \int_0^T (\langle x_0, f(t,x(t,x_0)), \int_{a(t)}^{b(t)} g(\tau,x(\tau,x_0))d\tau)\rangle)dt - c] \quad (32)$$

where $x^0(t,x_0)$ is the limiting function of the sequence of functions $x_m(t,x_0)$.

Since this function is approximation determined from the sequence of functions

$$\Delta_m(t,x_0) = \frac{1}{T\langle x_0,e\rangle}[\langle x_0\,,x_0\rangle + \int_0^T (\langle x_0, f(t,x_m(t,x_0)), \int_{a(t)}^{b(t)} g(\tau,x_m(\tau,x_0))d\tau)\rangle)dt - c] \quad (33)$$

Theorem4 . Let all assumptions and conditions of theorem 1.1 are satisfied, then the following inequality is satisfied :

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le \Lambda^{m+1}(E - \Lambda)^{-1}\|M(t)\| \ , \quad (34)$$

for all $m \ge 0$, $x_0 \in D_f$.

Proof: By the relations (32) and (33) we get

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le [(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)]\frac{1}{T}\int_0^T\|x(t,x_0) - x_m(t,x_0)\|\,dt$$

So that

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le [(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)]\frac{1}{T}\int_0^T \|M(t)\|\frac{T}{2}\Lambda^m(E - \Lambda)^{-1}\,dt$$
$$\le [\tfrac{T}{2}(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)]M\Lambda^m(E - \Lambda)^{-1}$$

But $\Lambda = \frac{T}{2}(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)$, thus the above inequality can be written as :

$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le \Lambda^{m+1}(E - \Lambda)^{-1}\|M(t)\|$, i.e.

The inequality (34) will be satisfied for all ≥ 0 .

Theorem 5 . Let the system (1) , (2) be defined on the interval $[a,b]$.

Suppose that for ≥ 0 , the function $\Delta_m(0,x_0)$ defined according to formula (33) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+h\le x\le b-h}\Delta_m(0,x_0) &\le -\Lambda^{m+1}(E - \Lambda)^{-1}\|M(t)\| \\ \max_{a+h\le x\le b-h}\Delta_m(0,x_0) &\ge \Lambda^{m+1}(E - \Lambda)^{-1}\|M(t)\| \end{aligned}\right\} \quad (35)$$

Then the system (1), (2) has a periodic solution $x = x(t,x_0)$ for which $x_0 \in [\,a+h\,,b-h]$.

where $h = \frac{T}{2}\ \|M(t)\|$

Proof. Let x_1 and x_2 be any two points in the interval $[a+h\,,b-h]$ such that

$$\left.\begin{aligned} \Delta_m(0,x_1) &= \min_{a+h\le x\le b-h}\Delta_m(0,x) \\ \Delta_m(0,x_2) &= \max_{a+h\le x\le b-h}\Delta_m(0,x)\ , \end{aligned}\right\} \quad (36)$$

Taking into account inequalities (34) and (35) we have , we have

$$\left.\begin{aligned} \Delta(0,x_1) &= \Delta_m(0,x_1) + [\Delta(0,x_1) - \Delta_m(0,x_1)] \le 0 \\ \Delta(0,x_2) &= \Delta_m(0,x_2) + [\Delta(0,x_2) - \Delta_m(0,x_2)] \ge 0 \end{aligned}\right\} \quad (37)$$

It follows from the inequalities (37) and the continuity of the function $\Delta(0,x_0)$,That there exists an isolated singular point x^0, $x^0 \in [x_1,x_2]$, such that $\Delta(0,x^0) = 0$. This means that the system (1), (2) has a periodic solution $x = x(t,x_0)$ for which $x_0 \in [\,a+h\,,b-h]$

Remark 1 (Samoilenko 1976)**.** Theorem **5** is proved when x_0 is a scalar singular point which should be isolated.

Theorem 6.. If the function $\Delta(0, x_0)$ is defined by $\Delta: D_f \to R^n$, and

$$\Delta(0,x_0) = \frac{1}{T\langle x_0,e\rangle}\int_0^T (\langle x_0, f(t,x(t,x_0), \int_{a(t)}^{b(t)} g(\tau,x(\tau,x_0))d\tau)\rangle)dt] \quad (38)$$

where $x^0(t,x_0)$ is a limit of the sequence functions (1.12) . then the following inequalities hold.

$$\|\Delta(0,x_0)\| \le \|M(t)\| \quad , \quad (39)$$

and

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \le \tfrac{2}{T}\Lambda(E-\Lambda)^{-1}\|x_0^1 - x_0^2\| , \quad (40)$$

For all x_0 , $x_0^1, x_0^2 \in D_f$ and E is identity matrix.

Proof. From the properties of the function $x^0(t,x_0)$ as in theorem 2 , the function $\Delta(0,x_0)$ is continuous and bounded by M in the domain $R^1 \times D_f$.

By using (39) ,we have

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \le \frac{1}{T}\int_0^T \left\| f\left(t, x(t,x_0^1), y(t,x_0^1)\right) - f\left(t,x(t,x_0^2),y(t,x_0^2)\right)\right\| dt$$

$$\le [\,(k_1 + k_2 LH)]\frac{1}{T}\int_0^T \|x(t,x_0^1) - x(t,x_0^2)\|\,dt$$

$$\le [\,(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\,]\|x(t,x_0^1) - x(t,x_0^2)\|$$

And hence,

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \le \tfrac{2}{T}\left[\tfrac{T}{2}\,(k_1 + k_2 LH)\right]\|x(t,x_0^1) - x(t,x_0^2)\|$$

$$\|\Delta(0,x_0^1) - \Delta(0,x_0^2)\| \le \tfrac{2}{T}\Lambda\|x(t,x_0^1) - x(t,x_0^2)\| \quad (41)$$

Where $x^0(t,x_0^1), x^0(t,x_0^2)$ are the solution of the integral equation:

$$x(t,x_0^k) = x_0^k + \int_0^t [f\left(s,x(s,x_0^k), \int_{a(t)}^{b(t)} g\left(\tau,x(\tau,x_0^k)\right)d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0 , f\left(s,x(s,x_0^k), \int_{a(t)}^{b(t)} g\left(\tau,x(\tau,x_0^k)\right)d\tau\right)\rangle ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] \quad (42)$$

with

$x_0^k(t,x_0) = x_0^k$, where $k = 1,2$.

From (42) , we get

$$\|x(t,x_0^1) - x(t,x_0^2)\| \le \|x_0^1 - x_0^2\| + [(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\,\alpha(t)\|x^0(s,x_0^1) - x^0(s,x_0^2)\|$$

$$\|x(t,x_0^1) - x(t,x_0^2)\| \le \|x_0^1 - x_0^2\| + \left[(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\,\tfrac{T}{2}\right]\|x^0(s,x_0^1) - x^0(s,x_0^2)\|$$

$$\|x(t,x_0^1) - x(t,x_0^2)\| \le \|x_0^1 - x_0^2\| + \Lambda\|x^0(s,x_0^1) - x^0(s,x_0^2)\|$$

$$(E-\Lambda)\|x(t,x_0^1) - x(t,x_0^2)\| \le \|x_0^1 - x_0^2\|$$

So that ,

$$\|x(t,x_0^1) - x(t,x_0^2)\| \le (E-\Lambda)^{-1}\|x_0^1 - x_0^2\| \quad (43)$$

By using (43) in (41), we get (40) .

Remark 2. (Mitropolsky1979)**.** Theorem 6 confirms the stability of the solution for the system (1), (2) ,that is when a slight change happens in the point x_0 ,then a slight change will happen in the function $\Delta(0,x_0)$.

5. EXISTENCE AND UNIQUENESS SOLUTION OF (1), (2)

In this section, we prove the existence uniqueness theorem of the problem (1), (2) using Banach fixed point theorem.

Theorem 7. (Banach Fixed Point Theorem). Let the vector functions $f(t,x,y)$ and $g(t,x)$ in the problem (1), (2) are defined and continuous on the domain (3) and satisfies all conditions of the theorem 2.1 , then the problem (1), (2) has a unique continuous on the domain (3) .

Proof . Let $(C[0,T^*], \|.\|)$ be Banach space and T^*be a mapping on $C[0,T]$ as follows

$$T^*x(t,x_0) = x_0 + \int_0^t [f\left(s,x(s,x_0), \int_{a(t)}^{b(t)} g(\tau,x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s,x(s,x_0), \int_{a(t)}^{b(t)} g(\tau,x(\tau,x_0))d\tau\right)$$

$$\rangle ds]ds + \frac{te}{T\langle x_0, e\rangle}[c - \langle x_0, x_0\rangle] \quad (44)$$

And

$$T^*w(t,x_0) = x_0 + \int_0^t [f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0, e\rangle}\int_0^T \langle x_0, f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right) \rangle ds]ds + \frac{te}{T\langle x_0, e\rangle}[c - \langle x_0, x_0\rangle] \quad (45)$$

Since $f(t,x,y)$ and $g(t,x)$ are continuous in the interval $[0,T]$ and x_0, y_0 are fixed points then

$$\int_0^t [f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0, e\rangle}\int_0^T \langle x_0, f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) \rangle ds]ds + \frac{te}{T\langle x_0, e\rangle}[c - \langle x_0, x_0\rangle]$$

And

$$\int_0^t [f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0, e\rangle}\int_0^T \langle x_0, f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right) \rangle ds]ds + \frac{te}{T\langle x_0, e\rangle}[c - \langle x_0, x_0\rangle]$$

are continuous functions on $C[0,T]$

Therefore $Tx(t,x_0), Ty(t,x_0) \in C[0,T]$

Let $x(t,x_0), y(t,x_0) \in C[0,T]$ then

$$\|T^*x(t,x_0) - T^*w(t,x_0)\| \le \Lambda\|x(t,x_0) - w(t,x_0)\| \quad (46)$$

Let $q(t) = \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau$

$$\|y(t,x_0) - q(t,x_0)\| = \left\|\int_{a(t)}^{b(t)} g(s, x(s))ds - \int_{a(t)}^{b(t)} w(s, x_0(s))ds\right\|$$

$$\|y(t,x_0) - q(t,x_0)\| \le L(t)\|x(t,x_0) - w(t,x_0)\|H \quad (47)$$

Now , taking

$$\|T^*x(t,x_0) - T^*w(t,x_0)\| = \max_{t\in[0,T]}\Big|x_0 + \int_0^t [f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right) - \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, x(s,x_0), \int_{a(t)}^{b(t)} g(\tau, x(\tau,x_0))d\tau\right)\rangle ds]ds + \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle] - x_0 - \int_0^t [f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right) + \frac{e}{T\langle x_0,e\rangle}\int_0^T \langle x_0, f\left(s, w(s,x_0), \int_{a(t)}^{b(t)} g(\tau, w(\tau,x_0))d\tau\right)\rangle ds]ds - \frac{te}{T\langle x_0,e\rangle}[c - \langle x_0,x_0\rangle]\Big|$$

$$\le \max_{t\in[0,T]}\Big|\int_0^t [f(s, x(s,x_0), y(s,x_0)) - f(s, w(s,x_0), q(s,x_0)) - \frac{e}{T}\int_0^T f(s, x(s,x_0), y(s,x_0)) - f(s, w(s,x_0), q(s,x_0))ds]ds\Big|$$

$$\le \max_{t\in[0,T]}\Big[\int_0^t K_1(t)|x(s,x_0) - w(s,x_0)| + K_2(t)L(t)H|x(s,x_0) - w(s,x_0)|ds - \frac{t}{T}\int_0^t K_1(t)|x(s,x_0) - w(s,x_0)| + K_2(t)L(t)H|x(s,x_0) - w(s,x_0)|ds + \frac{t}{T}\int_t^T K_1(t)|x(s,x_0) - w(s,x_0)| + K_2(t)L(t)H|x(s,x_0) - w(s,x_0)|\,ds\Big]$$

$$\le \max_{t\in[0,T]}\left[\left(t\left(1 - \frac{t}{T}\right) + \frac{t}{T}(T-t)\right)(K_1(t) + K_2(t)L(t)H)|x(s,x_0) - w(s,x_0)|\right]$$

$$\le \alpha(t)(\|K_1(t)\| + \|K_2(t)\|\|L(t)\|H)\max_{t\in[0,T]}|x(t,x_0) - w(t,x_0)|$$

$$\le \Lambda \max_{t\in[0,T]}|x(t,x_0) - w(t,x_0)|$$

Thus

$$\|T^*x(t,x_0) - T^*w(t,x_0)\| \le \Lambda\|x(t,x_0) - w(t,x_0)\|$$

T^* is a contraction mapping on $C[0,T]$ from theorem 2.1,we get

$T^*x(t,x_0) = x(t,x_0)$ and $T^*w(t,x_0) = w(t,x_0)$ are fixed point

and hence $\begin{pmatrix} x(t,x_0) \\ w(t,x_0)\end{pmatrix}$ is a unique continuous solution of (1), (2) .

CONCLUSION

The existence and approximation of the periodic solution of nonlinear integro-differential equation with nonlinear boundary condition is using by assuming the function $f(t,x,y)$ and$g(t,x,)$are measurable in t and bounded by Lebesgue integrable function which have the weaker conditions. The numerical – analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by Samoilenko (1976).

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